4. INVERSE TRIGONOMETRIC FUNCTIONS

INVERTIBLE FUNCTIONS *A one-one onto function is called an invertible function.*

INVERSE OF A FUNCTION Let $f : X \to Y$ be a one-one onto function. Then, for each $y \in Y$, there exists a unique element $x \in X$ such that $f(x) = y$.

So, we define a new function, denoted by f^{-1}, called the *inverse* of f, as

$$f^{-1} : Y \to X : f^{-1}(y) = x \;\Leftrightarrow\; f(x) = y.$$

Clearly,

$$\text{domain}\,(f^{-1}) = \text{range}\,(f) \text{ and } \text{range}\,(f^{-1}) = \text{domain}\,(f).$$

Trigonometric functions are, in general, not one-one onto.

Therefore, their inverses do not exist.

However, if we restrict their domains, we can make them one-one onto, enabling us to have their inverses.

EXAMPLE The sine function restricted to any of the intervals $[-\pi/2, \pi/2]$, $[3\pi/2, 5\pi/2]$, etc., is one-one onto and in each case, the range is $[-1, 1]$.

Therefore, we can define the inverse of the sine function, denoted by $\sin^{-1} x$, in each of these intervals. Corresponding to each such interval, we get a branch of $\sin^{-1} x$.

The branch $[-\pi/2, \pi/2]$ is called the *principal-value branch* and the value belonging to it is called the *principal value*.

We denote the inverse of the sine function as $\sin^{-1} x$, called *sine inverse x*.

$$\therefore\quad \sin^{-1} : [-1,\ 1] \to \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right] \text{ and } \sin^{-1} x = \theta \;\Leftrightarrow\; x = \sin\theta.$$

Thus, $\sin^{-1} x$ is a function whose domain is $[-1, 1]$ and range is $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$.

Note that $\sin^{-1} x$ does not mean $(\sin x)^{-1}$.

Similarly, we can define the principal-value branches of the remaining five trigonometrical functions.

The following table shows the inverse trigonometric functions and their principal-value branches.

Function	Domain	Range (Principal value)
1. $y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
2. $y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3. $y = \tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
4. $y = \text{cosec}^{-1} x$	$R - (-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5. $y = \sec^{-1} x$	$R - [-1, 1]$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
6. $y = \cot^{-1} x$	R	$(0, \pi)$

SOLVED EXAMPLES

EXAMPLE 1 *Find the principal value of each of the following:*

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ *(ii)* $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii) $\tan^{-1}(\sqrt{3})$ *(iv)* $\text{cosec}^{-1}(2)$

SOLUTION (i) We know that the range of the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$. Then,

$$\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence, the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

(ii) We know that the range of the principal-value branch of $\cos^{-1}$ is $[0, \pi]$.

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$. Then,

$$\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6} \in [0, \pi].$$

Hence, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

(iii) We know that the range of the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(\sqrt{3}) = \theta$. Then,

$$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}(\sqrt{3})$ is $\frac{\pi}{3}$.

(iv) We know that the range of the principal-value branch of cosec^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\text{cosec}^{-1}(2) = \theta$. Then,

$$\text{cosec}\,\theta = 2 = \text{cosec}\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}.$$

Hence, the principal value of $\text{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

EXAMPLE 2 *Find the principal value of each of the following:*

(i) $\sin^{-1}\left(\frac{-1}{2}\right)$ *(ii)* $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ *(iii)* $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

(iv) $\tan^{-1}(-1)$ *(v)* $\text{cosec}^{-1}(-2)$ *(vi)* $\tan^{-1}(-\sqrt{3})$

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{-1}{2}\right) = \theta$. Then,

$$\sin\theta = -\frac{1}{2} = \sin\left(\frac{-\pi}{6}\right), \text{ where } \frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence, the principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is $\frac{-\pi}{6}$.

(ii) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \theta$. Then,

$$\sin\theta = \frac{-\sqrt{3}}{2} = \sin\left(\frac{-\pi}{3}\right), \text{ where } \frac{-\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

Hence, the principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is $\frac{-\pi}{3}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \theta$. Then,

$$\tan\theta = \frac{-1}{\sqrt{3}} = \tan\left(\frac{-\pi}{6}\right), \text{ where } \frac{-\pi}{6} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\frac{-\pi}{6}$.

(iv) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(-1) = \theta$. Then,

$$\tan\theta = -1 = \tan\left(\frac{-\pi}{4}\right), \text{ where } \frac{-\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}(-1)$ is $\frac{-\pi}{4}$.

(v) We know that the principal-value branch of cosec^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\text{cosec}^{-1}(-2) = \theta$. Then,

$$\text{cosec}\,\theta = -2 = \text{cosec}\left(\frac{-\pi}{6}\right), \text{ where } \frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}.$$

Hence, the principal value of $\text{cosec}^{-1}(-2)$ is $\frac{-\pi}{6}$.

(vi) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(-\sqrt{3}) = \theta$. Then,

$$\tan\theta = -\sqrt{3} = \tan\left(\frac{-\pi}{3}\right), \text{ where } \frac{-\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Hence, the principal value of $\tan^{-1}(-\sqrt{3})$ is $\frac{-\pi}{3}$.

EXAMPLE 3 *Find the principal value of each of the following:*

(i) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ *(ii)* $\cos^{-1}\left(\frac{-1}{2}\right)$ *(iii)* $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

SOLUTION (i) We know that the range of the principal value of $\cos^{-1}$ is $[0, \pi]$.

Let $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \theta$. Then,

$$\cos\theta = \frac{-1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}.$$

$$\therefore \quad \theta = \frac{3\pi}{4} \in [0, \pi].$$

Hence, the principal value of $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

(ii) We know that the range of the principal value of $\cos^{-1}$ is $[0, \pi]$.

Let $\cos^{-1}\left(\frac{-1}{2}\right) = 0$. Then,

$$\cos 0 = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}.$$

$$\therefore \quad \theta = \frac{2\pi}{3} \in [0, \pi].$$

Hence, the principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is $\frac{2\pi}{3}$.

(iii) We know that the range of the principal value of $\cot^{-1}$ is $(0, \pi)$.

Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 0$. Then,

$$\cot 0 = \frac{-1}{\sqrt{3}} = -\cot\frac{\pi}{3} = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\frac{2\pi}{3}.$$

$$\therefore \quad \theta = \frac{2\pi}{3} \in (0, \pi).$$

Hence, the principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$.

EXAMPLE 4 *Find the principal value of each of the following:*

(i) $\cot^{-1}(-\sqrt{3})$ *(ii)* $\sec^{-1}(-\sqrt{2})$ *(iii)* $\operatorname{cosec}^{-1}(-1)$

SOLUTION (i) We know that the range of the principal value of $\cot^{-1}$ is $(0, \pi)$.

Let $\cot^{-1}(-\sqrt{3}) = \theta$. Then,

$$\cot\theta = -\sqrt{3} = -\cot\frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6}.$$

$$\therefore \quad \theta = \frac{5\pi}{6} \in (0,\ \pi).$$

Hence, the principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{5\pi}{6}$.

(ii) We know that the range of principal value of $\sec^{-1}$ is $[0,\ \pi] - \left\{\frac{\pi}{2}\right\}$.

Let $\sec^{-1}(-\sqrt{2}) = \theta$. Then,

$$\sec\theta = -\sqrt{2} = -\sec\frac{\pi}{4} = \sec\left(\pi - \frac{\pi}{4}\right) = \sec\frac{3\pi}{4}.$$

$$\therefore \quad \theta = \frac{3\pi}{4} \in [0,\ \pi] - \left\{\frac{\pi}{2}\right\}.$$

Hence, the principal value of $\sec^{-1}(-\sqrt{2})$ is $\frac{3\pi}{4}$.

(iii) We know that the range of principal value of $\operatorname{cosec}^{-1}$ is $\left[\frac{-\pi}{2},\ \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{cosec}^{-1}(-1) = \theta$. Then, $\operatorname{cosec}\theta = -1$.

$$\operatorname{cosec}\theta = -1 = -\operatorname{cosec}\frac{\pi}{2} = \operatorname{cosec}\left(\frac{-\pi}{2}\right).$$

$$\therefore \quad \theta = \frac{-\pi}{2} \in \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right] - \{0\}.$$

Hence, the principal value of $\operatorname{cosec}^{-1}(-1)$ is $\frac{-\pi}{2}$.

EXAMPLE 5 *Find the value of* $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$. **[CBSE 2012C]**

SOLUTION We know that the ranges of principal values of $\cos^{-1}$ and $\sin^{-1}$ are $[0,\ \pi]$ and $\left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]$ respectively.

Let $\cos^{-1}\left(\frac{1}{2}\right) = \theta_1$. Then,

$$\cos\theta_1 = \frac{1}{2} = \cos\frac{\pi}{3} \ \Rightarrow\ \theta_1 = \frac{\pi}{3} \in [0,\ \pi].$$

Let $\sin^{-1}\left(\frac{1}{2}\right) = \theta_2$. Then,

$$\sin\theta_2 = \frac{1}{2} = \sin\frac{\pi}{6} \ \Rightarrow\ \theta_2 = \frac{\pi}{6} \in \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right].$$

$$\therefore \quad \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \left(2 \times \frac{\pi}{6}\right) = \left(\frac{\pi}{3} + \frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

EXAMPLE 6 *Find the value of* $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$.

SOLUTION We know that the ranges of principal values of $\tan^{-1}$, $\cos^{-1}$ and $\sin^{-1}$ are $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $[0, \pi]$ and $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ respectively.

Let $\tan^{-1}(1) = \theta_1$. Then,

$$\tan\theta_1 = 1 = \tan\frac{\pi}{4} \Rightarrow \theta_1 = \frac{\pi}{4} \in [0, \pi].$$

Let $\cos^{-1}\left(\frac{-1}{2}\right) = \theta_2$. Then,

$$\cos\theta_2 = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}.$$

$$\therefore \quad \theta_2 = \frac{2\pi}{3} \in [0, \pi].$$

Let $\sin^{-1}\left(\frac{-1}{2}\right) = \theta_3$. Then,

$$\sin\theta_3 = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right) \Rightarrow \theta_3 = \frac{-\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

$$\therefore \quad \tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) = \left(\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}\right) = \frac{3\pi}{4}.$$

EXAMPLE 7 *Find the value of* $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

SOLUTION We know that the ranges of principal values of $\tan^{-1}$ and $\sec^{-1}$ are $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ respectively.

Let $\tan^{-1}\sqrt{3} = \theta_1$. Then,

$$\tan\theta_1 = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta_1 = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Let $\sec^{-1}(-2) = \theta_2$. Then,

$$\sec\theta_2 = -2 = -\sec\frac{\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}.$$

$$\therefore \quad \theta_2 = \frac{2\pi}{3} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}.$$

Hence, $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \left(\frac{\pi}{3} - \frac{2\pi}{3}\right) = \frac{-\pi}{3}$.

EXERCISE 4A

1. Find the principal value of:

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(\frac{1}{2}\right)$ (iii) $\cos^{-1}\left(\frac{1}{2}\right)$ (iv) $\tan^{-1}(1)$

(v) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (vi) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (vii) $\operatorname{cosec}^{-1}(\sqrt{2})$

2. Find the principal value of:

(i) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}(-\sqrt{3})$

(iv) $\sec^{-1}(-2)$ (v) $\operatorname{cosec}^{-1}(-\sqrt{2})$ (vi) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

3. Evaluate $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}$.

HINT: $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\theta \Rightarrow \cos\theta=\frac{-\sqrt{3}}{2}=-\cos\frac{\pi}{6}=\cos\left(\pi-\frac{\pi}{6}\right)=\cos\frac{5\pi}{6}\Rightarrow\theta=\frac{5\pi}{6}$.

$\therefore$ given expression $=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)=\cos\pi=-1$.

4. Evaluate $\sin\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right\}$.

HINT: Given exp. $=\sin\left\{\frac{\pi}{2}-\left(\frac{-\pi}{3}\right)\right\}=\sin\frac{5\pi}{6}=\sin\left(\pi-\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$.

ANSWERS (EXERCISE 4A)

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{6}$ (iii) $\frac{\pi}{3}$ (iv) $\frac{\pi}{4}$ (v) $\frac{\pi}{6}$ (vi) $\frac{\pi}{6}$ (vii) $\frac{\pi}{4}$

2. (i) $\frac{-\pi}{4}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{-\pi}{3}$ (iv) $\frac{2\pi}{3}$ (v) $\frac{-\pi}{4}$ (vi) $\frac{2\pi}{3}$

3. -1 **4.** $\frac{1}{2}$

PROPERTIES OF INVERSE FUNCTIONS

THEOREM 1 *Prove that:*

(i) $\sin^{-1}(\sin x) = x,\ x \in \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos x) = x,\ \ x \in [0,\ \pi]$

(iii) $\tan^{-1}(\tan x) = x,\ \ x \in \left(\frac{-\pi}{2},\ \frac{\pi}{2}\right)$

(iv) $\text{cosec}^{-1}(\text{cosec}\, x) = x,\ \ x \in \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right] - \{0\}$

(v) $\sec^{-1}(\sec x) = x,\ \ x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot x) = x,\ \ x \in (0, \pi)$

PROOF (i) Let $\sin x = y$, where $x \in \left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]$.

Then, $\sin^{-1} y = x \ \Rightarrow\ \sin^{-1}(\sin x) = x \qquad [\because\ y = \sin x]$.

$\therefore\ \sin^{-1}(\sin x) = x.$

(ii) Let $\cos x = y$, where $x \in [0,\ \pi]$.

Then, $\cos^{-1} y = x \ \Rightarrow\ \cos^{-1}(\cos x) = x \qquad [\because\ y = \cos x]$.

$\therefore\ \cos^{-1}(\cos x) = x.$

Similarly, the other results may be proved.

THEOREM 2 *Prove that:*

(i) $\sin(\sin^{-1} x) = x,\ \ x \in [-1,\ 1]$

(ii) $\cos(\cos^{-1} x) = x,\ \ x \in [-1,\ 1]$

(iii) $\tan(\tan^{-1} x) = x,\ \ x \in R$

(iv) $\text{cosec}(\text{cosec}^{-1} x) = x,\ \ x \in R - [-1, 1]$

(v) $\sec(\sec^{-1} x) = x,\ \ x \in R - [-1,\ 1]$

(vi) $\cot(\cot^{-1} x) = x,\ \ x \in R$

PROOF (i) Let $\sin^{-1} x = \theta$, where $x \in [-1,\ 1]$.

Then, $\sin\theta = x \ \Rightarrow\ \sin(\sin^{-1} x) = x \qquad [\because\ \theta = \sin^{-1} x]$.

$\therefore\ \sin(\sin^{-1} x) = x.$

(ii) Let $\cos^{-1} x = \theta$, where $x \in [-1,\ 1]$.

Then, $\cos\theta = x \ \Rightarrow\ \cos(\cos^{-1} x) = x \qquad [\because\ \theta = \cos^{-1} x]$.

$\therefore\ \cos(\cos^{-1} x) = x.$

Similarly, the other results may be proved.

THEOREM 3 *Prove that:*

(i) $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x, (x \geq 1 \text{ or } x \leq -1)$

(ii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x, (x \geq 1 \text{ or } x \leq -1)$

(iii) $\tan^{-1}\frac{1}{x} = \cot^{-1}x, \ (x > 0)$

PROOF (i) $\sin^{-1}\frac{1}{x} = \theta \Rightarrow \sin\theta = \frac{1}{x}$

$\Rightarrow \operatorname{cosec}\theta = x$

$\Rightarrow 0 = \operatorname{cosec}^{-1}x$

$\Rightarrow \sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x.$

Hence, $\sin^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}x.$

(ii) $\cos^{-1}\frac{1}{x} = 0 \Rightarrow \cos 0 = \frac{1}{x}$

$\Rightarrow \sec 0 = x$

$\Rightarrow \theta = \sec^{-1}x$

$\Rightarrow \cos^{-1}\frac{1}{x} = \sec^{-1}x.$

Hence, $\cos^{-1}\frac{1}{x} = \sec^{-1}x.$

(iii) $\tan^{-1}\frac{1}{x} = \theta \Rightarrow \tan\theta = \frac{1}{x}$

$\Rightarrow \cot\theta = x$

$\Rightarrow 0 = \cot^{-1}x$

$\Rightarrow \tan^{-1}\frac{1}{x} = \cot^{-1}x.$

Hence, $\tan^{-1}\frac{1}{x} = \cot^{-1}x.$

THEOREM 4 *Prove that:*

(i) $\sin^{-1}(-x) = -\sin^{-1}x, \ x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x, \ x \in R$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x, \ |x| \geq 1$

PROOF (i) Let $\sin^{-1}(-x) = 0$. Then,

$\sin^{-1}(-x) = \theta \Rightarrow -x = \sin\theta$

$\Rightarrow x = -\sin 0 = \sin(-0)$

$\Rightarrow -0 = \sin^{-1}x$

$\Rightarrow \theta = -\sin^{-1}x$

$\Rightarrow \sin^{-1}(-x) = -\sin^{-1}x.$

$\therefore \quad \sin^{-1}(-x) = -\sin^{-1} x.$

Similarly, the other results may be proved.

THEOREM 5 *Prove that:*

(i) $\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x, \quad |x| \geq 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad x \in R$

PROOF (i) Let $\cos^{-1}(-x) = \theta$. Then,

$$\cos^{-1}(-x) = \theta \Rightarrow -x = \cos\theta$$
$$\Rightarrow x = -\cos\theta = \cos(\pi - \theta)$$
$$\Rightarrow \cos^{-1} x = (\pi - \theta) = \pi - \cos^{-1}(-x)$$
$$\Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1} x.$$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x.$

Similarly, the other results may be proved.

THEOREM 6 *Prove that:*

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad x \in R$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, \quad |x| \geq 1$

PROOF (i) Let $\sin^{-1} x = \theta$. Then,

$$\sin^{-1} x = \theta \Rightarrow x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$$
$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$
$$\Rightarrow \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

$\therefore \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$

Similarly, the other results may be proved.

THEOREM 7 *Prove that:*

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{ if } xy > -1$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } |x| < 1$

PROOF (i) Let $xy < 1$, $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$. Then,

$$\tan\theta = x \quad \text{and} \quad \tan\phi = y$$

$$\Rightarrow \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{x + y}{1 - xy}$$

$$\Rightarrow \theta + \phi = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right).$$

$$\therefore \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right).$$

(ii) Let $xy > -1$.

On replacing y by $-y$ in (i), we get

$$\tan^{-1} x + \tan^{-1}(-y) = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$$

$$\Rightarrow \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right).$$

(iii) Let $|x| < 1$.

Replacing y by x in (i), we get

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right).$$

THEOREM 8 *Prove that:*

(i) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right), \quad |x| < 1$

(ii) $2\tan^{-1} x = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right), \quad |x| \geq 0$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right), \quad |x| < 1$

PROOF Let $\tan^{-1} x = \theta$. Then, $x = \tan\theta$.

(i) $\sin^{-1}\left(\frac{2x}{1 + x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right) \qquad [\because \quad x = \tan\theta]$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1} x.$$

$$\therefore 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right).$$

(ii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ $[\because \quad x = \tan\theta]$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta = 2\tan^{-1} x.$

$\therefore \quad 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$

(iii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$ $[\because \quad x = \tan\theta]$

$= \tan^{-1}(\tan 2\theta)$

$= 2\theta = 2\tan^{-1} x.$

$\therefore \quad 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$

THEOREM 9 *Prove that* $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1-x^2}], \ \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}.$

PROOF Let $\sin^{-1} x = \theta$. Then, $x = \sin\theta$.

$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta$

$= 2\sin\theta \cdot \sqrt{(1-\sin^2\theta)}$

$= 2x\sqrt{1-x^2}$

$\Rightarrow \quad 2\theta = \sin^{-1}\{2x\sqrt{1-x^2}\} \Rightarrow \quad 2\sin^{-1} x = \sin^{-1}\{2x\sqrt{1-x^2}\}.$

Hence, $2\sin^{-1} x = \sin^{-1}\{2x\sqrt{1-x^2}\}.$

THEOREM 10 *Prove that* $2\cos^{-1} x = \cos^{-1}(2x^2 - 1), \ \frac{1}{\sqrt{2}} \le x \le 1.$

PROOF Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$.

$\therefore \quad \cos 2\theta = (2\cos^2\theta - 1) = (2x^2 - 1)$

$\Rightarrow \quad 2\theta = \cos^{-1}(2x^2 - 1)$

$\Rightarrow \quad 2\cos^{-1} x = \cos^{-1}(2x^2 - 1) \qquad [\because \quad \theta = \cos^{-1} x].$

Hence, $2\cos^{-1} x = \cos^{-1}(2x^2 - 1).$

THEOREM 11 *Prove that* $3\sin^{-1} x = \sin^{-1}(3x - 4x^3), \ x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$

PROOF Put $\sin^{-1} x = \theta$. Then, $x = \sin\theta$.

Now, $\sin 3\theta = (3\sin\theta - 4\sin^3\theta) = (3x - 4x^3)$

$\Rightarrow \quad 3\theta = \sin^{-1}(3x - 4x^3) \Rightarrow \quad 3\sin^{-1} x = \sin^{-1}(3x - 4x^3) \ [\because \quad \theta = \sin^{-1} x].$

Hence, $3\sin^{-1} x = \sin^{-1}(3x - 4x^3).$

THEOREM 12 *Prove that* $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x),\ x \in \left[\frac{1}{2},\ 1\right]$.

PROOF Put $\cos^{-1}x = \theta$. Then, $x = \cos\theta$.

Now, $\cos 3\theta = (4\cos^3\theta - 3\cos\theta) = (4x^3 - 3x)$

$\Rightarrow\ 3\theta = \cos^{-1}(4x^3 - 3x) \Rightarrow\ 3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$ [$\because\ \theta = \cos^{-1}x$].

Hence, $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$.

THEOREM 13 *Prove that:*

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{(1-x^2)(1-y^2)}\}$

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\{xy + \sqrt{(1-x^2)(1-y^2)}\}$

PROOF (i) Let $\sin^{-1}x = \theta_1$, and $\sin^{-1}y = \theta_2$. Then,

$\sin\theta_1 = x$ and $\sin\theta_2 = y$

$\Rightarrow\ \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

$= \{\sin\theta_1 \cdot \sqrt{1 - \sin^2\theta_2}\} + \{\sqrt{1 - \sin^2\theta_1} \cdot \sin\theta_2\}$

$= \{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\Rightarrow\ (\theta_1 + \theta_2) = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$\Rightarrow\ \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$.

Hence, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$.

The other results can be proved similarly.

THEOREM 14 *Prove that:*

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}(\sqrt{1+x^2}) = \cot^{-1}\left(\frac{1}{x}\right)$

PROOF (i) Let $\sin^{-1} x = \theta$. Then, $\sin \theta = x$.

$$\therefore \quad \cos \theta = \sqrt{1 - x^2}, \quad \tan \theta = \frac{x}{\sqrt{1 - x^2}}, \quad \text{cosec } \theta = \frac{1}{x},$$

$$\sec \theta = \frac{1}{\sqrt{1 - x^2}} \quad \text{and} \quad \cot \theta = \frac{\sqrt{1 - x^2}}{x}.$$

$$\therefore \quad \theta = \cos^{-1} \sqrt{1 - x^2}, \quad \theta = \tan^{-1} \left(\frac{x}{\sqrt{1 - x^2}} \right), \quad \theta = \text{cosec}^{-1} \frac{1}{x},$$

$$\theta = \sec^{-1} \left(\frac{1}{\sqrt{1 - x^2}} \right) \quad \text{and} \quad \theta = \cot^{-1} \left(\frac{\sqrt{1 - x^2}}{x} \right).$$

$$\text{Hence, } \sin^{-1} x = \cos^{-1} \sqrt{1 - x^2} = \tan^{-1} \left(\frac{x}{\sqrt{1 - x^2}} \right)$$

$$= \text{cosec}^{-1} \frac{1}{x} = \sec^{-1} \left(\frac{1}{\sqrt{1 - x^2}} \right)$$

$$= \cot^{-1} \left(\frac{\sqrt{1 - x^2}}{x} \right).$$

Similarly, the other results can be proved.

SOLVED EXAMPLES

EXAMPLE 1 *Find the value of:*

(i) $\sin^{-1} \left(\sin \frac{\pi}{3} \right)$ *(ii)* $\cos^{-1} \left(\cos \frac{2\pi}{3} \right)$ *(iii)* $\tan^{-1} \left(\tan \frac{\pi}{4} \right)$

SOLUTION We have

(i) $\sin^{-1} \left(\sin \frac{\pi}{3} \right) = \frac{\pi}{3}$, since $\frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2} \right]$.

(ii) $\cos^{-1} \left(\cos \frac{2\pi}{3} \right) = \frac{2\pi}{3}$, since $\frac{2\pi}{3} \in [0, \pi]$.

(iii) $\tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4}$, since $\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2} \right)$.

EXAMPLE 2 *Find the value of:*

(i) $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$ *(ii)* $\cos^{-1} \left(\cos \frac{7\pi}{6} \right)$ *(iii)* $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad \sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$.

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$

$= \sin^{-1}\left(\sin\frac{\pi}{3}\right)$ [$\because \sin(\pi - \theta) = \sin\theta$]

$= \frac{\pi}{3} \quad \left\{\because \quad \frac{\pi}{3} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}$.

Hence, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

(ii) We know that the principal-value branch of $\cos^{-1}$ is $[0, \pi]$.

$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$

$= \cos^{-1}\left\{\cos\frac{5\pi}{6}\right\}$ {$\because \cos(2\pi - \theta) = \cos\theta$}

$= \frac{5\pi}{6}$.

Hence, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$.

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\}$

$= \tan^{-1}\left\{-\tan\frac{\pi}{4}\right\} \quad \left[\because \tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4}\right]$

$= \tan^{-1}\left\{\tan\left(\frac{-\pi}{4}\right)\right\} \quad [\because \quad -\tan\theta = \tan(-\theta)]$

$= \frac{-\pi}{4}$.

Hence, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{-\pi}{4}$.

EXAMPLE 3 *Find the value of:*

(i) $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ *(ii)* $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ *(iii)* $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad \sin^{-1}\left(\sin\frac{3\pi}{5}\right) \neq \frac{3\pi}{5}$.

Now, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{2\pi}{5}\right)\right\}$

$= \sin^{-1}\left(\sin\frac{2\pi}{5}\right)$ $[\because \sin(\pi - \theta) = \sin\theta]$

$= \frac{2\pi}{5}$ $\left\{\because \quad \frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}$.

Hence, $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$.

(ii) We know that the principal-value branch of $\cos^{-1}$ is $[0, \pi]$.

$\therefore \quad \cos^{-1}\left(\cos\frac{13\pi}{6}\right) \neq \frac{13\pi}{6}$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi + \frac{\pi}{6}\right)\right\}$

$= \cos^{-1}\left\{\cos\frac{\pi}{6}\right\}$ $[\because \cos(2\pi + \theta) = \cos\theta]$

$= \frac{\pi}{6}$.

Hence, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \quad \tan^{-1}\left(\tan\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left\{\tan\left(\pi + \frac{\pi}{6}\right)\right\}$

$= \tan^{-1}\left(\tan\frac{\pi}{6}\right)$ $[\because \tan(\pi + \theta) = \tan\theta]$

$= \frac{\pi}{6}$.

Hence, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\frac{\pi}{6}$.

EXAMPLE 4 *(i) Show that* $\sin^{-1}\left\{\sin\frac{3\pi}{4}\right\}\neq\frac{3\pi}{4}$ *and find its value.*

(ii) Show that $\cos^{-1}\left\{\cos\left(\frac{-\pi}{3}\right)\right\}\neq\frac{-\pi}{3}$ *and find its value.*

(iii) Show that $\tan^{-1}\left\{\tan\frac{5\pi}{6}\right\}\neq\frac{5\pi}{6}$ *and find its value.*

SOLUTION (i) We know that the principal-value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$.

$\therefore\ \sin^{-1}\left\{\sin\frac{3\pi}{4}\right\}\neq\frac{3\pi}{4}$.

Now, $\sin^{-1}\left\{\sin\frac{3\pi}{4}\right\}=\sin^{-1}\left\{\sin\left(\pi-\frac{\pi}{4}\right)\right\}$

$$=\sin^{-1}\left\{\sin\frac{\pi}{4}\right\}\left[\because\ \sin\left(\pi-\frac{\pi}{4}\right)=\sin\frac{\pi}{4}\right]$$

$$=\frac{\pi}{4}\left\{\because\ \frac{\pi}{4}\in\left[\frac{-\pi}{2},\ \frac{\pi}{2}\right]\right\}.$$

$\therefore\ \sin^{-1}\left\{\sin\frac{3\pi}{4}\right\}=\frac{\pi}{4}$.

(ii) We know that the principal-value branch of $\cos^{-1}$ is $[0,\ \pi]$.

$\therefore\ \cos^{-1}\left\{\cos\left(\frac{-\pi}{3}\right)\right\}\neq\frac{-\pi}{3}$.

Now, $\cos^{-1}\left\{\cos\left(\frac{-\pi}{3}\right)\right\}=\cos^{-1}\left\{\cos\frac{\pi}{3}\right\}$ $[\because\ \cos(-\theta)=\cos\theta]$

$$=\frac{\pi}{3}\left[\because\ \frac{\pi}{3}\in[0,\ \pi]\right].$$

$\therefore\ \cos^{-1}\left\{\cos\left(\frac{-\pi}{3}\right)\right\}=\frac{\pi}{3}$.

(iii) We know that the principal-value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

$\therefore\ \tan^{-1}\left\{\tan\frac{5\pi}{6}\right\}\neq\frac{5\pi}{6}$.

Now, $\tan^{-1}\left\{\tan\frac{5\pi}{6}\right\}=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{6}\right)\right\}$

$$=\tan^{-1}\left\{\tan\left(\frac{-\pi}{6}\right)\right\}$$

$[\because\ \tan(\pi-\theta)=\tan(-\theta)]$

$$= \frac{-\pi}{6} \quad \left[\because \ \frac{-\pi}{6} \in \left(\frac{-\pi}{2}, \ \frac{\pi}{2}\right)\right].$$

$$\therefore \ \tan^{-1}\left\{\tan\frac{5\pi}{6}\right\} = \frac{-\pi}{6}.$$

EXAMPLE 5 *Evaluate:*

(i) $\sin\left(\cos^{-1}\frac{3}{5}\right)$ *(ii)* $\cos\left(\tan^{-1}\frac{3}{4}\right)$

SOLUTION (i) Let $\cos^{-1}\frac{3}{5} = \theta$, where $\theta \in [0, \ \pi]$.

Then, $\cos\theta = \frac{3}{5}$.

Since $\theta \in [0, \ \pi]$, we have $\sin\theta > 0$.

$$\therefore \ \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Hence, $\sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\theta = \frac{4}{5}$.

(ii) Let $\tan^{-1}\frac{3}{4} = \theta$, where $\theta \in \left(\frac{-\pi}{2}, \ \frac{\pi}{2}\right)$.

Then, $\tan\theta = \frac{3}{4}$.

Since $\theta \in \left(\frac{-\pi}{2}, \ \frac{\pi}{2}\right)$, so $\cos\theta > 0$.

$$\therefore \ \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + \frac{9}{16}}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Hence, $\cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\theta = \frac{4}{5}$.

EXAMPLE 6 *Evaluate:*

(i) $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\}$ **[CBSE 2008]** *(ii)* $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

(iii) $\sin(\cot^{-1} x)$ *(iv)* $\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right)$

SOLUTION (i) We know that $\sin^{-1}(-\theta) = -\sin^{-1}\theta$.

$$\therefore \ \sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\} = \sin\left\{\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right\}$$

$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) \left[\because \ \sin^{-1}\frac{1}{2} = \frac{\pi}{6}\right]$$

$$= \sin\frac{\pi}{2} = 1.$$

(ii) Let $\cos^{-1}\frac{4}{5} = \theta$, where $\theta \in [0, \pi]$.

Then, $\cos\theta = \frac{4}{5}$.

Since $\theta \in [0, \pi] \Rightarrow \frac{1}{2}\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin\frac{1}{2}\theta > 0$.

$$\therefore \sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \sin\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(\frac{4}{5}\right)}{2}} = \frac{1}{\sqrt{10}}.$$

(iii) Let $\cot^{-1}x = \theta$. Then, $\theta \in [0, \pi]$.

$\therefore \sin(\cot^{-1}x) = \sin\theta > 0$.

Now, $\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}}$.

$$\therefore \sin(\cot^{-1}x) = \sin\theta = \frac{1}{\sqrt{1+x^2}}.$$

(iv) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$, where $\theta \in [0, \pi]$.

$$\therefore \tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right) = \tan\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{(1-\sqrt{5}/3)}{(1+\sqrt{5}/3)}}$$

$$= \sqrt{\frac{(3-\sqrt{5})}{(3+\sqrt{5})} \times \frac{(3-\sqrt{5})}{(3-\sqrt{5})}} = \frac{(3-\sqrt{5})}{2}.$$

EXAMPLE 7 *Evaluate* $\sin\left[2\cos^{-1}\left(\frac{-3}{5}\right)\right]$.

SOLUTION Let $\cos^{-1}\left(\frac{-3}{5}\right) = \theta$, where $\theta \in [0, \pi]$.

Then, $\cos\theta = \frac{-3}{5}$.

Since $\theta \in [0, \pi]$, we have $\sin\theta > 0$.

$$\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

$$\therefore \sin\left[2\cos^{-1}\left(\frac{-3}{5}\right)\right] = \sin 2\theta$$

$$= 2\sin\theta\cos\theta = \left\{2 \times \frac{4}{5} \times \left(\frac{-3}{5}\right)\right\} = \frac{-24}{25}.$$

EXAMPLE 8 *Evaluate* $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$.

SOLUTION Let $\sin^{-1}\frac{3}{5}=A$ and $\sin^{-1}\frac{5}{13}=B$. Then,

$$A, B \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos A > 0 \text{ and } \cos B > 0.$$

$$\therefore \sin A = \frac{3}{5} \text{ and } \sin B = \frac{5}{13}$$

$$\Rightarrow \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{and } \cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}.$$

$$\therefore \cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right) = \cos(A+B)$$

$$= \cos A \cos B - \sin A \sin B$$

$$= \left(\frac{4}{5}\times\frac{12}{13}\right) - \left(\frac{3}{5}\times\frac{5}{13}\right)$$

$$= \left(\frac{48}{65}-\frac{15}{65}\right) = \frac{33}{65}.$$

EXAMPLE 9 *Find the value of* $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$.

SOLUTION We have

$$\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = \tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\} \left[\because \sin^{-1}\frac{1}{2}=\frac{\pi}{6}\right]$$

$$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\}$$

$$= \tan^{-1}\left(2\times\frac{1}{2}\right) = \tan^{-1}1 = \frac{\pi}{4}.$$

EXAMPLE 10 *If* $\tan^{-1}\frac{4}{3}=\theta$, *find the value of* $\cos\theta$.

SOLUTION $\tan^{-1}\frac{4}{3}=\theta$, where $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \tan\theta = \frac{4}{3}.$$

We know that $\cos\theta > 0$, when $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\frac{16}{9}}} = \frac{3}{5}.$$

EXAMPLE 11 *If* $\cot^{-1}\left(-\frac{1}{5}\right) = \theta$, *find the value of* $\sin\theta$.

SOLUTION Given: $\cot^{-1}\left(-\frac{1}{5}\right) = \theta$, where $\theta \in (0, \pi)$.

$$\therefore \quad \cot\theta = \frac{-1}{5}.$$

$\sin\theta > 0$ in $(0, \pi)$.

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+\frac{1}{25}}} = \frac{5}{\sqrt{26}}.$$

EXAMPLE 12 *Prove that* $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$.

SOLUTION We know that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \quad xy < 1.$

Let $x = \frac{1}{7}$ and $y = \frac{1}{13}$. Then, $xy = \frac{1}{91} < 1$.

$$\therefore \quad \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\left\{\frac{\left(\frac{1}{7}+\frac{1}{13}\right)}{1-\frac{1}{7}\times\frac{1}{13}}\right\} = \tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\frac{2}{9}.$$

EXAMPLE 13 *Prove that* $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$. **[CBSE 2009, '12C]**

SOLUTION We have

$$\text{LHS} = \left\{\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right\} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left\{\frac{\left(\frac{3}{4}+\frac{3}{5}\right)}{\left(1-\frac{3}{4}\times\frac{3}{5}\right)}\right\} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\frac{\left(\frac{27}{11}-\frac{8}{19}\right)}{\left(1+\frac{27}{11}\times\frac{8}{19}\right)} = \tan^{-1}\left(\frac{425}{425}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 14 *Prove that* $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

[CBSE 2008, '09C]

SOLUTION We have

$$\text{LHS} = \left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}\right)+\left(\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{\left(\frac{1}{3}+\frac{1}{5}\right)}{\left(1-\frac{1}{3}\times\frac{1}{5}\right)}+\tan^{-1}\frac{\left(\frac{1}{7}+\frac{1}{8}\right)}{\left(1-\frac{1}{7}\times\frac{1}{8}\right)}$$

$$= \tan^{-1}\frac{\left(\frac{8}{15}\right)}{\left(\frac{14}{15}\right)}+\tan^{-1}\frac{\left(\frac{15}{56}\right)}{\left(\frac{55}{56}\right)}$$

$$= \tan^{-1}\frac{8}{14}+\tan^{-1}\frac{15}{55} = \tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11}$$

$$= \tan^{-1}\frac{\left(\frac{4}{7}+\frac{3}{11}\right)}{\left(1-\frac{4}{7}\times\frac{3}{11}\right)} = \tan^{-1}\frac{\left(\frac{65}{77}\right)}{\left(\frac{65}{77}\right)} = \tan^{-1}1 = \frac{\pi}{4} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 15 *Prove that* $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$.

SOLUTION We have

$$\text{LHS} = 2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\left(2\times\frac{1}{2}\right)}{\left\{1-\left(\frac{1}{2}\right)^2\right\}}+\tan^{-1}\frac{1}{7}\left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7} = \tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\left(\frac{4}{3}+\frac{1}{7}\right)}{\left(1-\frac{4}{3}\times\frac{1}{7}\right)} = \tan^{-1}\frac{\left(\frac{31}{21}\right)}{\left(\frac{17}{21}\right)} = \tan^{-1}\frac{31}{17} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 16 *Prove that* $2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right) = \tan^{-1}\frac{4}{3}$.

SOLUTION We have

$$\text{LHS} = 2\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{2}{9}$$

$$= \tan^{-1}\frac{\left(2\times\frac{1}{4}\right)}{\left\{1-\left(\frac{1}{4}\right)^2\right\}} + \tan^{-1}\frac{\left(2\times\frac{2}{9}\right)}{\left\{1-\left(\frac{2}{9}\right)^2\right\}}$$

$$= \tan^{-1}\frac{\left(\frac{1}{2}\right)}{\left(\frac{15}{16}\right)} + \tan^{-1}\frac{\left(\frac{4}{9}\right)}{\left(\frac{77}{81}\right)}$$

$$= \tan^{-1}\left(\frac{1}{2}\times\frac{16}{15}\right) + \tan^{-1}\left(\frac{4}{9}\times\frac{81}{77}\right)$$

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{36}{77}$$

$$= \tan^{-1}\frac{\left(\frac{8}{15}+\frac{36}{77}\right)}{\left(1-\frac{8}{15}\times\frac{36}{77}\right)} = \tan^{-1}\frac{(616+540)}{(1155-288)}$$

$$= \tan^{-1}\left(\frac{1156}{867}\right) = \tan^{-1}\frac{4}{3} = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 17 *Prove that* $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$. **[CBSE 2008C, '14]**

SOLUTION We have

$$\text{LHS} = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$

$$= \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\frac{\left(\frac{1}{7}+\frac{1}{8}\right)}{\left(1-\frac{1}{7}\times\frac{1}{8}\right)} + \tan^{-1}\frac{1}{18} = \tan^{-1}\frac{\left(\frac{15}{56}\right)}{\left(\frac{55}{56}\right)} + \tan^{-1}\frac{1}{18}$$

$$= \left(\tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18}\right) = \tan^{-1}\frac{\left(\frac{3}{11}+\frac{1}{18}\right)}{\left(1-\frac{3}{11}\times\frac{1}{18}\right)}$$

$$= \tan^{-1}\frac{\left(\frac{65}{198}\right)}{\left(\frac{195}{198}\right)} = \tan^{-1}\frac{1}{3} = \cot^{-1} 3 = \text{RHS}.$$

$\therefore$ LHS = RHS.

EXAMPLE 18 *Prove that* $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$. **[CBSE 2012]**

SOLUTION Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$. Then,

$$\sin x = \frac{3}{5} \text{ and } \sin y = \frac{8}{17}.$$

$$\therefore \quad \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{and } \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}.$$

$$\therefore \quad \cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$= \left(\frac{4}{5}\times\frac{15}{17}\right) + \left(\frac{3}{5}\times\frac{8}{17}\right) = \left(\frac{12}{17}+\frac{24}{85}\right) = \frac{84}{85}$$

$$\Rightarrow \quad x - y = \cos^{-1}\left(\frac{84}{85}\right) \quad \Rightarrow \quad \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}.$$

EXAMPLE 19 *Prove that* $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}$. **[CBSE 2011C]**

SOLUTION LHS $= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$, where $xy = \left(\frac{1}{4}\times\frac{2}{9}\right) = \frac{1}{18} < 1$

$$= \tan^{-1}\left\{\frac{\left(\frac{1}{4}+\frac{2}{9}\right)}{\left(1-\frac{1}{4}\times\frac{2}{9}\right)}\right\} = \tan^{-1}\left(\frac{17}{34}\right) = \tan^{-1}\frac{1}{2}.$$

Now, let $\frac{1}{2}\cos^{-1}\frac{3}{5} = \theta$. Then, $\cos 2\theta = \frac{3}{5}$.

$$\therefore \quad \tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{\left(1-\frac{3}{5}\right)}{\left(1+\frac{3}{5}\right)}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{2} = \theta \Rightarrow \quad \tan^{-1}\frac{1}{2} = \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

$$\text{Hence, } \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

EXAMPLE 20 *Prove that* $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$.

SOLUTION Let $\cos^{-1}\frac{4}{5} = 0$. Then, $\cos 0 = \frac{4}{5}$.

$$\therefore \quad \tan 0 = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-\frac{16}{25}}}{(4/5)} = \left(\frac{3}{5}\times\frac{5}{4}\right) = \frac{3}{4}$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{3}{4}.$$

$$\therefore \quad \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}.$$

$$\therefore \quad \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right) = \tan^{-1}\frac{27}{11}.$$

EXAMPLE 21 *Solve* $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$. **[CBSE 2008, '09, '12C, '13C]**

SOLUTION $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4} \quad \Rightarrow \quad \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{5x}{(1-6x^2)} = \tan\frac{\pi}{4} = 1 \quad \Rightarrow \quad 1-6x^2 = 5x$$

$$\Rightarrow \quad 6x^2+5x-1=0 \Rightarrow \quad 6x^2+6x-x-1=0$$

$$\Rightarrow \quad 6x(x+1)-(x+1)=0 \Rightarrow \quad (x+1)(6x-1)=0$$

$$\Rightarrow \quad x=-1 \quad \text{or} \quad x=\frac{1}{6}.$$

$$\Rightarrow \quad x=\frac{1}{6} \qquad [\because \quad x=-1 \text{ makes LHS negative}].$$

EXAMPLE 22 *Solve* $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$. **[CBSE 2008, '12C]**

SOLUTION We have

$$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{\left\{\frac{(x-1)}{(x-2)}+\frac{(x+1)}{(x+2)}\right\}}{\left\{1-\frac{(x-1)}{(x-2)}\cdot\frac{(x+1)}{(x+2)}\right\}}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x^2-4)-(x^2-1)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left\{\frac{(x^2+x-2)+(x^2-x-2)}{-3}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2-4}{-3}=\tan\frac{\pi}{4}=1 \Rightarrow 2x^2-4=-3$$

$$\Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}.$$

EXAMPLE 23 *Solve* $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$. **[CBSE 2008C]**

SOLUTION We have

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left\{\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right\}=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{2-x^2}\right)=\tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan\left\{\tan^{-1}\left(\frac{2x}{2-x^2}\right)\right\}=\frac{8}{31} \Rightarrow \frac{2x}{2-x^2}=\frac{8}{31}$$

$$\Rightarrow 8x^2+62x-16=0 \Rightarrow 4x^2+31x-8=0$$

$$\Rightarrow (4x-1)(x+8)=0 \Rightarrow x=\frac{1}{4} \text{ or } x=-8.$$

But, $x=-8$ gives LHS $=\tan^{-1}(-7)+\tan^{-1}(-9)$, which is negative, while RHS is positive. So, $x=-8$ is not possible.

Hence, $x=\frac{1}{4}$.

EXAMPLE 24 *Solve* $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, (x>0)$. **[CBSE 2008C, '11, '14C]**

SOLUTION We have

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, (x>0)$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \frac{3}{2}\tan^{-1}x=\tan^{-1}1=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}x=\left(\frac{\pi}{4}\times\frac{2}{3}\right)=\frac{\pi}{6} \Rightarrow x=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}.$$

Hence, $x=\frac{1}{\sqrt{3}}$.

EXAMPLE 25 *Solve* $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$.

SOLUTION We have

$$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)\right]=2\operatorname{cosec} x$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x}=2\operatorname{cosec} x \Rightarrow \cos x=\sin x$$

$$\Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}.$$

PROBLEMS BASED ON TRIGONOMETRIC FORMULAE

EXAMPLE 26 *Prove that* $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\frac{x}{2}, \ x<\pi.$

SOLUTION We have

$$\text{LHS}=\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}}$$

$$=\tan^{-1}\left(\tan\frac{x}{2}\right)=\frac{x}{2}=\text{RHS}.$$

$$\therefore \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\frac{x}{2}.$$

EXAMPLE 27 *Prove that* $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \left(\frac{\pi}{4} - x\right), \quad x < \pi.$

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) \text{ [dividing num. and denom. by } \cos x\text{]}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - x\right)\right\} = \left(\frac{\pi}{4} - x\right) = \text{RHS}.$$

$$\therefore \quad \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \left(\frac{\pi}{4} - x\right).$$

EXAMPLE 28 *Prove that* $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \left(\frac{\pi}{4} - \frac{x}{2}\right).$ **[CBSE 2012]**

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = \left(\frac{\pi}{4} - \frac{x}{2}\right) = \text{RHS}.$$

$$\therefore \quad \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \left(\frac{\pi}{4} - \frac{x}{2}\right).$$

EXAMPLE 29 *Prove that* $\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right) = \left(\frac{\pi}{4} + \frac{x}{2}\right), x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$ **[CBSE 2012]**

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \left(\frac{\pi}{4}+\frac{x}{2}\right) = \text{RHS}.$$

Hence, $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \left(\frac{\pi}{4}+\frac{x}{2}\right)$.

EXAMPLE 30 *Prove that* $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = 3\tan^{-1}\frac{x}{a}$.

SOLUTION Putting $x = a\tan\theta$, we get

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\frac{x}{a}.$$

Hence, $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = 3\tan^{-1}\frac{x}{a}$.

EXAMPLE 31 *Prove that* $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x.$

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \tan^{-1}\left\{\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x - a\sin x}{b\cos x}}\right\}$$

[on dividing num. and denom. by $b\cos x$]

$$= \tan^{-1}\left\{\frac{\frac{a}{b} - \tan x}{1 - \frac{a}{b}\tan x}\right\} = \tan^{-1}\left(\frac{p-q}{1-pq}\right), \text{where } \frac{a}{b} = p \text{ and } \tan x = q$$

$$= \tan^{-1} p - \tan^{-1} q = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$$

$$= \left(\tan^{-1}\frac{a}{b} - x\right) = \text{RHS}.$$

Hence, $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x - a\sin x}\right) = \left(\tan^{-1}\frac{a}{b} - x\right)$.

EXAMPLE 32 *Prove that* $\cot^{-1}\left\{\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right\} = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right)$.

[CBSE 2011, '14]

SOLUTION We have

$$\text{LHS} = \cot^{-1}\left\{\frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})}{(\sqrt{1+\sin x} - \sqrt{1-\sin x})} \times \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})}{(\sqrt{1+\sin x} + \sqrt{1-\sin x})}\right\}$$

$$= \cot^{-1}\left\{\frac{(1+\sin x) + (1-\sin x) + 2\sqrt{1-\sin^2 x}}{(1+\sin x) - (1-\sin x)}\right\}$$

$$= \cot^{-1}\left\{\frac{2(1+\cos x)}{2\sin x}\right\} = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left\{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right\} = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 33 *Prove that* $\tan^{-1}\left\{\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right\} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$. **[CBSE 2011, '14]**

SOLUTION Putting $x = \cos 0$, we get

$$\text{LHS} = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)} - \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)} + \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}\right\} = \tan^{-1}\left\{\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{0}{2} + \sin\frac{0}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right\} \qquad \left[\text{dividing num. and denom. by } \cos\frac{\theta}{2}\right]$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} = \left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 34 *Prove that* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x.$

SOLUTION Putting $x = \tan\theta$, we get

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right\} = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

$$\therefore \quad \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x.$$

EXAMPLE 35 *Prove that* $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\frac{x}{a}.$

SOLUTION Putting $x = a\sin\theta$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) = \tan^{-1}(\tan\theta)$$

$$= \theta = \sin^{-1}\frac{x}{a} = \text{RHS}.$$

$$\therefore \quad \tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) = \sin^{-1}\frac{x}{a}.$$

EXAMPLE 36 *Prove that* $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, $x \in [0, 1]$. **[CBSE 2010]**

SOLUTION Putting $x = \tan^2\theta$, we get

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \left(\frac{1}{2}\times 2\theta\right) = \theta = \tan^{-1}\sqrt{x} = \text{LHS}$$

$$[\because \quad x = \tan^2\theta \Rightarrow \tan\theta = \sqrt{x} \Rightarrow \theta = \tan^{-1}\sqrt{x}].$$

Hence, $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$.

EXAMPLE 37 *Prove that* $\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \left(\frac{\pi}{2} - \sec^{-1}x\right)$.

SOLUTION Putting $x = \sec\theta$, we get

$$\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{\sec^2\theta - 1}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}(\cot\theta)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\} = \left(\frac{\pi}{2} - \theta\right) = \left(\frac{\pi}{2} - \sec^{-1}x\right).$$

$$\therefore \quad \tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \left(\frac{\pi}{2} - \sec^{-1}x\right).$$

EXAMPLE 38 *Prove that* $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$

SOLUTION Putting $x^2=\cos 2\theta$, we get

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

[dividing num. and denom. by $\cos\theta$]

$$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}=\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

$$\therefore\quad \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

EXAMPLE 39 *Prove that* $2\tan^{-1}\frac{1}{x}=\sin^{-1}\left(\frac{2x}{x^2+1}\right).$

SOLUTION Let $\tan^{-1}\frac{1}{x}=\theta$. Then, $\frac{1}{x}=\tan\theta \;\Rightarrow\; x=\cot\theta$.

$$\therefore\quad \text{LHS}=2\tan^{-1}\frac{1}{x}=2\theta.$$

$$\text{RHS}=\sin^{-1}\left(\frac{2\cot\theta}{\cot^2\theta+1}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}(\sin 2\theta)=2\theta.$$

$\therefore$ LHS = RHS.

Hence, $2\tan^{-1}\frac{1}{x}=\sin^{-1}\left(\frac{2x}{x^2+1}\right).$

EXAMPLE 40 *Prove that* $\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}=\left(\frac{x+y}{1-xy}\right),$

where $|x|<1,\ y>0$ *and* $xy<1$. **[CBSE 2013]**

SOLUTION Putting $x = \tan\theta$ and $y = \tan\phi$, we get

$$\text{LHS} = \tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

$$= \tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)\right\}$$

$$= \tan\left\{\frac{1}{2}\sin^{-1}(\sin 2\theta) + \frac{1}{2}\cos^{-1}(\cos 2\phi)\right\}$$

$$= \tan\left\{\left(\frac{1}{2}\times 2\theta\right) + \left(\frac{1}{2}\times 2\phi\right)\right\}$$

$$= \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{(x+y)}{(1-xy)} = \text{RHS}.$$

$$\therefore \quad \tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1-x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\} = \frac{x+y}{1-xy}.$$

EXAMPLE 41 *Prove that* $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0.$

SOLUTION We have

$$\text{LHS} = \tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= (\tan^{-1}a - \tan^{-1}b) + (\tan^{-1}b - \tan^{-1}c) + (\tan^{-1}c - \tan^{-1}a)$$

$$= 0 = \text{RHS}.$$

$\therefore$ LHS = RHS.

SOME MORE EXAMPLES

EXAMPLE 1 *Prove that* $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}.$ **[CBSE 2010, '12]**

SOLUTION We have

$$\text{LHS} = \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$$

$$= \cos^{-1}\left\{\left(\frac{4}{5}\times\frac{12}{13}\right) - \sqrt{1-\left(\frac{4}{5}\right)^2}\cdot\sqrt{1-\left(\frac{12}{13}\right)^2}\right\}$$

$$= \cos^{-1}\left\{\frac{48}{65} - \sqrt{1-\left(\frac{16}{25}\right)}\cdot\sqrt{1-\frac{144}{169}}\right\}$$

$$= \cos^{-1}\left\{\frac{48}{65} - \sqrt{\frac{9}{25}} \cdot \sqrt{\frac{25}{169}}\right\}$$

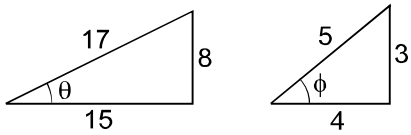
$$= \cos^{-1}\left\{\frac{48}{65} - \frac{3}{5} \times \frac{5}{13}\right\} = \cos^{-1}\left(\frac{48}{65} - \frac{15}{65}\right)$$

$$= \cos^{-1}\left(\frac{33}{65}\right) = \text{RHS}.$$

Hence, $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

EXAMPLE 2 *Prove that* $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$. **[CBSE 2014C]**

SOLUTION $\text{LHS} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$



$$= \cos^{-1}\frac{15}{17} + \cos^{-1}\frac{4}{5}$$

$$= \cos^{-1}\left\{\left(\frac{15}{17} \times \frac{4}{5}\right) - \sqrt{1 - \left(\frac{15}{17}\right)^2} \cdot \sqrt{1 - \left(\frac{4}{5}\right)^2}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \sqrt{1 - \left(\frac{225}{289}\right)} \cdot \sqrt{1 - \frac{16}{25}}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \sqrt{\frac{64}{289}} \cdot \sqrt{\frac{9}{25}}\right\} = \cos^{-1}\left\{\frac{12}{17} - \frac{8}{17} \times \frac{3}{5}\right\}$$

$$= \cos^{-1}\left\{\frac{12}{17} - \frac{24}{85}\right\} = \cos^{-1}\left(\frac{36}{85}\right) = \text{RHS}.$$

Hence, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$.

EXAMPLE 3 *Prove that* $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$. **[CBSE 2011C]**

SOLUTION We have

$$\text{LHS} = \tan^{-1}\frac{3}{4} - \tan\frac{17}{31}$$

$$= \tan^{-1}\left\{\frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2}\right\} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31} = \tan^{-1}\left\{\frac{\left(\frac{24}{7}-\frac{17}{31}\right)}{1+\left(\frac{24}{7}\times\frac{17}{31}\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{625}{217}\times\frac{217}{625}\right\} = \tan^{-1}1 = \frac{\pi}{4} = \text{RHS}.$$

Hence, $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$.

EXAMPLE 4 *Prove that* $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = \frac{\pi}{4}$. **[CBSE 2014]**

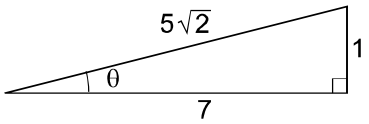
SOLUTION Let $\sec^{-1}\frac{5\sqrt{2}}{7} = \theta$. Then, $\sec\theta = \frac{5\sqrt{2}}{7}$. So, $\cos\theta = \frac{7}{5\sqrt{2}}$.

Draw a right triangle with base = 7 units

and hypotenuse $= 5\sqrt{2}$ units.

$\therefore$ perpendicular $= \sqrt{(5\sqrt{2})^2 - 7^2} = \sqrt{50-49} = 1$ unit.

$\therefore$ $\tan\theta = \frac{1}{7} \Rightarrow \theta = \tan^{-1}\frac{1}{7}$.



$\therefore$ $\text{LHS} = 2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{7}$

$$= 2\cdot\tan^{-1}\left\{\frac{\left(\frac{1}{5}+\frac{1}{8}\right)}{\left(1-\frac{1}{5}\times\frac{1}{8}\right)}\right\} + \tan^{-1}\frac{1}{7}$$

$$= 2\tan^{-1}\left(\frac{13}{40}\times\frac{40}{30}\right) + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right\} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\left(\frac{3}{4}+\frac{1}{7}\right)}{\left(1-\frac{3}{4}\times\frac{1}{7}\right)} = \tan^{-1}\frac{25}{25} = \tan^{-1}1$$

$$= \frac{\pi}{4} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 5 *Prove that* $\cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3 - 3x^2}\right) = \frac{\pi}{3}$. **[CBSE 2014C]**

SOLUTION Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$ and $\sqrt{1 - x^2} = \sin\theta$.

$$\therefore \quad \text{LHS} = \cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1 - x^2}\right)$$

$$= \theta + \cos^{-1}\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)$$

$$= \theta + \cos^{-1}\left\{\cos\left(\frac{\pi}{3} - \theta\right)\right\} = \theta + \left(\frac{\pi}{3} - \theta\right) = \frac{\pi}{3} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 6 *Prove that* $\cos\left[\tan^{-1}\{\sin(\cot^{-1} x)\}\right] = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$. **[CBSE 2010]**

SOLUTION Let $\cot^{-1} x = \theta$. Then, $x = \cot\theta$.

$$\therefore \quad \text{cosec}\,\theta = \sqrt{1 + \cot^2\theta} = \sqrt{1 + x^2}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{\sqrt{1 + x^2}}$$

$$\Rightarrow \quad \sin(\cot^{-1} x) = \frac{1}{\sqrt{1 + x^2}} \quad [\because \theta = \cot^{-1} x]$$

$$\Rightarrow \quad \tan^{-1}\{\sin(\cot^{-1} x)\} = \tan^{-1}\frac{1}{\sqrt{1 + x^2}} = \phi \text{ (say)}$$

$$\Rightarrow \quad \cos\left[\tan^{-1}\{\sin(\cot^{-1} x\}\right] = \cos\phi. \qquad \text{... (i)}$$

Now, $\tan^{-1}\frac{1}{\sqrt{1 + x^2}} = \phi \Rightarrow \tan\phi = \frac{1}{\sqrt{1 + x^2}}$

$$\Rightarrow \quad \sec\phi = \sqrt{1 + \tan^2\phi} = \sqrt{1 + \frac{1}{(1 + x^2)}} = \sqrt{\frac{2 + x^2}{1 + x^2}}$$

$$\Rightarrow \quad \cos\phi = \sqrt{\frac{1 + x^2}{2 + x^2}}. \qquad \text{... (ii)}$$

From (i) and (ii), we get

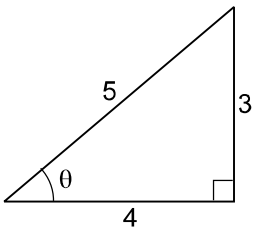
$$\cos\left[\tan^{-1}\{\sin(\cot^{-1} x)\}\right] = \sqrt{\frac{x^2 + 1}{x^2 + 2}}.$$

EXAMPLE 7 *Prove that* $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$. **[CBSE 2012]**

SOLUTION Let $\sin^{-1}\frac{3}{5} = \theta$. Then, $\sin\theta = \frac{3}{5}$.

Draw a right triangle in which

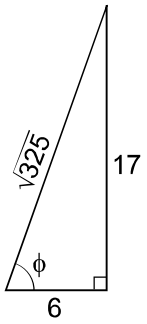
perp. = 3 units,
hyp. = 5 units and
base $= \sqrt{(5)^2 - (3)^2} = \sqrt{16} = 4$ units.



$\therefore \quad \tan 0 = \frac{3}{4} \Rightarrow 0 = \tan^{-1}\frac{3}{4}.$

$$\therefore \quad \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \cos\left(0 + \tan^{-1}\frac{2}{3}\right)$$

$$= \cos\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) = \cos\left[\tan^{-1}\left\{\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right\}\right]$$



$$= \cos\left(\tan^{-1}\frac{17}{6}\right) = \cos\left\{\cos^{-1}\frac{6}{\sqrt{325}}\right\} = \frac{6}{5\sqrt{13}}$$

$$\left[\because \tan\phi = \frac{17}{6} \Rightarrow \cos\phi = \frac{6}{\sqrt{325}} = \frac{6}{5\sqrt{13}} \Rightarrow \phi = \cos^{-1}\frac{6}{5\sqrt{13}} \right.$$
$$\left. \Rightarrow \cos\phi = \frac{6}{5\sqrt{13}}\right].$$

EXAMPLE 8 *Evaluate:*

(i) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$ *(ii)* $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

SOLUTION (i) Let $\cos^{-1}\frac{\sqrt{5}}{3} = 0$. Then, $\cos 0 = \frac{\sqrt{5}}{3}.$

$$\therefore \quad \tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\} = \tan\frac{0}{2}$$

$$= \sqrt{\frac{1-\cos 0}{1+\cos 0}} = \sqrt{\frac{\left(1-\frac{\sqrt{5}}{3}\right)}{\left(1+\frac{\sqrt{5}}{3}\right)}} = \sqrt{\frac{(3-\sqrt{5})}{(3+\sqrt{5})}}$$

$$= \sqrt{\frac{(3-\sqrt{5})}{(3+\sqrt{5})}\times\frac{(3-\sqrt{5})}{(3+\sqrt{5})}} = \frac{(3-\sqrt{5})}{\sqrt{9-5}}$$

$$= \frac{3-\sqrt{5}}{\sqrt{4}} = \frac{(3-\sqrt{5})}{2}.$$

(ii) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

$$= \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)-\tan^{-1}1\right\}$$

$$= \tan\left\{\tan^{-1}\frac{5}{12}-\tan^{-1}1\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}.$$

EXAMPLE 9 *Prove that* $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}.$

SOLUTION Let $\cos^{-1}\frac{a}{b}=0$. Then, $\frac{a}{b}=\cos 0$.

$$\therefore \quad \text{LHS} = \tan\left\{\frac{\pi}{4}+\frac{1}{2}\theta\right\} = \tan\left\{\frac{\pi}{4}-\frac{1}{2}\theta\right\}$$

$$= \frac{1+\tan\left(\frac{0}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}+\frac{1-\tan\left(\frac{0}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}$$

$$= \frac{\left\{1+\tan\left(\frac{0}{2}\right)\right\}^2+\left\{1-\tan\left(\frac{0}{2}\right)\right\}^2}{\left\{1-\tan^2\left(\frac{\theta}{2}\right)\right\}}$$

$$= 2\left\{\frac{1+\tan^2\left(\frac{0}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}\right\} = \frac{2}{\cos 0} = \frac{2b}{a} = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 10 *Prove that* $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}.$ **[CBSE 2011]**

SOLUTION We have

$$\text{LHS} = \frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\cos^{-1}\frac{1}{3} = \frac{9}{4}\cdot\sin^{-1}\sqrt{1-\frac{1}{9}}$$

$$= \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \text{RHS}.$$

Hence, LHS = RHS.

EXAMPLE 11 *Find the value of* $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$. **[CBSE 2011]**

SOLUTION We have

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left\{\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right\}$$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left\{\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}(1)\right\}$$

$$= \tan^{-1}(1) = \frac{\pi}{4}.$$

EXAMPLE 12 *Prove that* $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5} = \frac{1}{2}\sin^{-1}\frac{4}{5}$. **[CBSE 2010C]**

SOLUTION We have

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\left\{\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\times\frac{2}{9}\right)}\right\} = \tan^{-1}\frac{1}{2}.$$

Now, $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow \tan^{-1}x = \frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\Rightarrow \tan^{-1}\frac{1}{2} = \frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right) = \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

And, $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow \tan^{-1}x = \frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow \tan^{-1}\frac{1}{2} = \frac{1}{2}\sin^{-1}\left(\frac{2\times\frac{1}{2}}{1+\frac{1}{4}}\right) = \frac{1}{2}\sin^{-1}\frac{4}{5}.$$

Hence, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5} = \frac{1}{2}\sin^{-1}\frac{4}{5}$.

Problems on Trigonometric Equations

EXAMPLE 13 *Solve for x:* $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$. **[CBSE 2009, '10C]**

SOLUTION We have

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}(2\operatorname{cosec} x \cot x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow 2\operatorname{cosec} x \cot x = 2\operatorname{cosec} x \Rightarrow \cot x = 1 \Rightarrow x = \frac{\pi}{4}.$$

Hence, $x = \frac{\pi}{4}$.

EXAMPLE 14 *Solve for x:* $2\tan^{-1}(\sin x) = \tan^{-1}(2\operatorname{cosec} x)$, $x \neq \frac{\pi}{2}$. **[CBSE 2012]**

SOLUTION We have

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan\,(2\sec x)$$

$$\Rightarrow \tan^{-1}(2\sec x \tan x) = \tan^{-1}(2\sec x)$$

$$\Rightarrow 2\sec x \tan x = 2\sec x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}.$$

Hence, $x = \frac{\pi}{4}$.

EXAMPLE 15 *If* $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$, *find the value of x.* **[CBSE 2014]**

SOLUTION We have

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1} 1 = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1} x = \left(\frac{\pi}{2} - \sin^{-1}\frac{1}{5}\right)$$

$$\Rightarrow \cos^{-1} x = \cos^{-1}\frac{1}{5} \Rightarrow x = \frac{1}{5}.$$

Hence, $x = \frac{1}{5}$.

EXAMPLE 16 *If* $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$ *then find the value of x.*

SOLUTION The given equation may be written as

$$\sin^{-1}\frac{x}{5}+\sin^{-1}\frac{4}{5}=\frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{5}=\left(\frac{\pi}{2}-\sin\frac{4}{5}\right)$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{5}=\cos^{-1}\frac{4}{5}=\sin^{-1}\sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$=\sin^{-1}\sqrt{1-\frac{16}{25}}=\sin^{-1}\frac{3}{5}$$

$$\Rightarrow \quad \frac{x}{5}=\frac{3}{5} \Rightarrow \quad x=3.$$

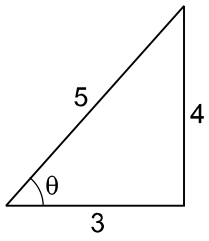
Hence, $x = 3$.

EXAMPLE 17 *Solve for x:* $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$. **[CBSE 2014]**

SOLUTION Let $\cot^{-1}\frac{3}{4}=\theta$. Then, $\cot\theta=\frac{3}{4}$.

Draw a right triangle in which base = 3 units
and perpendicular = 4 units.

Then, hypotenuse $=\sqrt{3^2+4^2}=\sqrt{25}=5$ units.



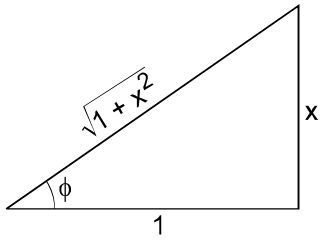
$$\therefore \quad \sin\theta=\frac{4}{5} \Rightarrow \quad \theta=\sin^{-1}\frac{4}{5}.$$

So, $\sin\left(\cot^{-1}\frac{3}{4}\right)=\sin\left(\sin^{-1}\frac{4}{5}\right)=\frac{4}{5}$.

Let $\tan^{-1}x=\phi$. Then, $\tan\phi=\frac{x}{1}$.

Draw a right triangle in which perpendicular $= x$ and base = 1.

Then, hypotenuse $=\sqrt{1+x^2}$.



$$\therefore \quad \cos\phi=\frac{1}{\sqrt{1+x^2}}.$$

So, $\cos(\tan^{-1}x)=\cos\phi=\frac{1}{\sqrt{1+x^2}}$.

Thus, $\frac{1}{\sqrt{1+x^2}}=\frac{4}{5} \Rightarrow \quad \frac{1}{(1+x^2)}=\frac{16}{25} \Rightarrow \quad 16x^2=9$

$$\Rightarrow \quad x^2=\frac{9}{16} \Rightarrow \quad x=\pm\frac{3}{4}.$$

Hence, $x=\pm\frac{3}{4}$.

EXAMPLE 18 *If* $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, *find the value of x.* **[CBSE 2014]**

SOLUTION The given equation may be written as

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)=\frac{\pi}{4}-\tan^{-1}\left(\frac{x+2}{x+4}\right)$$

$$=\tan^{-1}1-\tan^{-1}\left(\frac{x+2}{x+4}\right)\quad\left[\because\quad\frac{\pi}{4}=\tan^{-1}1\right]$$

$$=\tan^{-1}\left\{\frac{1-\frac{x+2}{x+4}}{1+\frac{x+2}{x+4}}\right\}=\tan^{-1}\left(\frac{2}{2x+6}\right)$$

$$=\tan^{-1}\left(\frac{1}{x+3}\right).$$

$$\therefore\quad\frac{x-2}{x-4}=\frac{1}{x+3}\;\Rightarrow\;(x-2)(x+3)=(x-4)$$

$$\Rightarrow\;x^2+x-6=x-4\;\Rightarrow\;x^2=2\;\Rightarrow\;x=\pm\sqrt{2}.$$

Hence, $x=\pm\sqrt{2}$.

EXAMPLE 19 *Solve for x:* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3},\ -1<x<1.$ **[CBSE 2011C]**

SOLUTION The given equation may be written as

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

$$\Rightarrow\;2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

$$\Rightarrow\;\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{6}$$

$$\Rightarrow\;\frac{2x}{1-x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\;(1-x^2)=2\sqrt{3}x$$

$$\Rightarrow\;x^2+2\sqrt{3}x-1=0$$

$$\Rightarrow\;x=\frac{-2\sqrt{3}\pm\sqrt{12+4}}{2}=\frac{-2\sqrt{3}\pm4}{2}$$

$$=(-\sqrt{3}+2)\text{ or }(-\sqrt{3}-2).$$

But, $-1<x<1$. So, $x\neq(-\sqrt{3}-2)$.

Hence, $x=(2-\sqrt{3})$.

EXERCISE 4B

Very-Short-Answer Questions

Find the principal value of each of the following:

1. $\sin^{-1}\left(\frac{-1}{2}\right)$ 2. $\cos^{-1}\left(\frac{-1}{2}\right)$ 3. $\tan^{-1}(-1)$
4. $\sec^{-1}(-2)$ 5. $\operatorname{cosec}^{-1}(-\sqrt{2})$ 6. $\cot^{-1}(-1)$
7. $\tan^{-1}(-\sqrt{3})$ 8. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ 9. $\operatorname{cosec}^{-1}(2)$

Find the principal value of each of the following:

10. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ 11. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ 12. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$
13. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ 14. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ 15. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$
16. $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\}$ 17. $\cot(\tan^{-1}x + \cot^{-1}x)$
18. $\operatorname{cosec}(\sin^{-1}x + \cos^{-1}x)$ 19. $\sin(\sec^{-1}x + \operatorname{cosec}^{-1}x)$
20. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}$ 21. $\tan^{-1}1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$
22. $\sin^{-1}\left\{\sin\frac{3\pi}{5}\right\}$ **[CBSE 2009]**

ANSWERS (EXERCISE 4B)

1. $\frac{-\pi}{6}$ **2.** $\frac{2\pi}{3}$ **3.** $\frac{-\pi}{4}$ **4.** $\frac{2\pi}{3}$ **5.** $\frac{5\pi}{4}$ **6.** $\frac{3\pi}{4}$ **7.** $\frac{-\pi}{3}$ **8.** $\frac{5\pi}{6}$
9. $\frac{\pi}{6}$ **10.** $\frac{\pi}{3}$ **11.** $\frac{-\pi}{4}$ **12.** $\frac{5\pi}{6}$ **13.** $\frac{\pi}{6}$ **14.** $\frac{\pi}{6}$ **15.** $\frac{5\pi}{6}$ **16.** 1
17. 0 **18.** 1 **19.** 1 **20.** $\frac{2\pi}{3}$ **21.** $\frac{3\pi}{4}$ **22.** $\frac{2\pi}{5}$

HINTS TO THE GIVEN QUESTIONS (EXERCISE 4B)

1. Range of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = \theta \Rightarrow \sin\theta = -\frac{1}{2} = \sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow \quad \theta = \frac{-\pi}{6}.$

2. Range of $\cos^{-1}$ is $[0, \pi]$.

$\therefore \quad \cos^{-1}\left(\frac{-1}{2}\right) = \theta \Rightarrow \cos\theta = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\Rightarrow \quad \theta = \frac{2\pi}{3}.$

3. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \quad \tan^{-1}(-1) = 0 \Rightarrow \tan 0 = -1 = -\tan\frac{\pi}{4} = \tan\left(\frac{-\pi}{4}\right)$

$\Rightarrow \quad \theta = \frac{-\pi}{4}.$

4. Range of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\sec^{-1}(-2) = \theta \Rightarrow \sec\theta = -2 = -\sec\frac{\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}$

$\Rightarrow \quad \theta = \frac{2\pi}{3}.$

5. Range of cosec^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

$\therefore \quad \text{cosec}^{-1}(-\sqrt{2}) = \theta \Rightarrow \text{cosec}\,\theta = -\sqrt{2} = -\text{cosec}\frac{\pi}{4} = \text{cosec}\left(\pi + \frac{\pi}{4}\right) = \text{cosec}\frac{5\pi}{4}$

$\Rightarrow \quad \theta = \frac{5\pi}{4}.$

6. Range of $\cot^{-1}$ is $(0, \pi)$.

$\therefore \quad \cot^{-1}(-1) = \theta \Rightarrow \cot\theta = -1 = -\cot\frac{\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = \cot\frac{3\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}.$

7. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \quad \tan^{-1}(-\sqrt{3}) = \theta \Rightarrow \tan\theta = -\sqrt{3} = \tan\left(\frac{-\pi}{3}\right) \Rightarrow \theta = \frac{-\pi}{3}.$

8. Range of $\sec^{-1}$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right) = \theta \Rightarrow \sec\theta = \frac{-2}{\sqrt{3}} = -\sec\frac{\pi}{6} = \sec\left(\pi - \frac{\pi}{6}\right) = \sec\frac{5\pi}{6}$

$\Rightarrow \quad \theta = \frac{5\pi}{6}.$

9. Range of cosec^{-1} is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

$\therefore \quad \text{cosec}^{-1}(2) = \theta \Rightarrow \text{cosec}\,\theta = 2 = \text{cosec}\frac{\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{6}.$

10. Range of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \quad \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$$

$$= \sin^{-1}\left\{\sin\frac{\pi}{3}\right\} = \frac{\pi}{3}.$$

11. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\}$$

$$= \tan^{-1}\left\{-\tan\frac{\pi}{4}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{-\pi}{4}\right)\right\} = \frac{-\pi}{4}.$$

12. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left\{\cos\frac{5\pi}{6}\right\} = \frac{5\pi}{6}.$$

13. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi + \frac{\pi}{6}\right)\right\}$$

$$= \cos^{-1}\left\{\cos\frac{\pi}{6}\right\} = \frac{\pi}{6}.$$

14. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left\{\tan\left(\pi + \frac{\pi}{6}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\pi}{6}\right\} = \frac{\pi}{6}.$$

15. $\tan^{-1}\sqrt{3} = \theta_1 \quad \Rightarrow \quad \tan\theta_1 = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta_1 = \frac{\pi}{3}.$

$$\cot^{-1}(-\sqrt{3}) = \theta_2 \quad \Rightarrow \quad \cot\theta_2 = -\sqrt{3} = -\cot\frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6}$$

$$\Rightarrow \quad \theta_2 = \frac{5\pi}{6}.$$

$$\therefore \quad \tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \left(\frac{\pi}{3} - \frac{5\pi}{6}\right)$$

$$= \frac{-3\pi}{6} = \frac{-\pi}{2}.$$

16. Let $\sin^{-1}\left(-\frac{1}{2}\right)=\theta$. Then, $\sin\theta=-\frac{1}{2}=\sin\left(-\frac{\pi}{6}\right) \Rightarrow \theta=\frac{-\pi}{6}$.

$\therefore$ given exp. $=\sin\left[\frac{\pi}{3}-\left(\frac{-\pi}{6}\right)\right]=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\frac{3\pi}{6}=\sin\frac{\pi}{2}=1.$

17. $(\tan^{-1}a+\cot^{-1}a)=\frac{\pi}{2} \Rightarrow \cot(\tan^{-1}a+\cot^{-1}a)=\cot\frac{\pi}{2}=0.$

18. $(\sin^{-1}a+\cos^{-1}a)=\frac{\pi}{2} \Rightarrow \operatorname{cosec}(\sin^{-1}a+\cos^{-1}a)=\operatorname{cosec}\frac{\pi}{2}=1.$

19. $(\sec^{-1}a+\operatorname{cosec}^{-1}a)=\frac{\pi}{2} \Rightarrow \sin(\sec^{-1}a+\operatorname{cosec}^{-1}a)=\sin\frac{\pi}{2}=1.$

20. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=\left(\frac{\pi}{3}+2\times\frac{\pi}{6}\right)=\left(\frac{\pi}{3}+\frac{\pi}{3}\right)=\frac{2\pi}{3}$.

21. $\tan^{-1}1+\cos^{-1}\left(\frac{-1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)=\left\{\frac{\pi}{4}+\frac{2\pi}{3}+\left(\frac{-\pi}{6}\right)\right\}=\left(\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}\right)$

$$=\frac{(3\pi+8\pi-2\pi)}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}.$$

22. See hint for Q. 10.

EXERCISE 4C

1. Prove that:

   (i) $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x,\ x<1$

   (ii) $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$

2. Prove that

   $$\sin^{-1}(2x\sqrt{1-x^2})=2\sin^{-1}x,\ |x|\le\frac{1}{\sqrt{2}}.$$ **[CBSE 2008]**

3. Prove that:

   (i) $\sin^{-1}(3x-4x^3)=3\sin^{-1}x,\ |x|\le\frac{1}{2}$

   (ii) $\cos^{-1}(4x^3-3x)=3\cos^{-1}x,\ \frac{1}{2}\le x\le 1$

   (iii) $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=3\tan^{-1}x,\ |x|<\frac{1}{\sqrt{3}}$

   (iv) $\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ **[CBSE 2010]**

4. Prove that:

   (i) $\cos^{-1}(1-2x^2)=2\sin^{-1}x$

   (ii) $\cos^{-1}(2x^2-1)=2\cos^{-1}x$

(iii) $\sec^{-1}\left(\frac{1}{2x^2-1}\right) = 2\cos^{-1} x$

(iv) $\cot^{-1}(\sqrt{1+x^2} - x) = \frac{\pi}{2} - \frac{1}{2}\cot^{-1} x$ **[CBSE 2007]**

5. Prove that:

(i) $\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}\right) = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{y}$

(ii) $\tan^{-1}\left(\frac{x+\sqrt{x}}{1-x^{3/2}}\right) = \tan^{-1} x + \tan^{-1}\sqrt{x}$

(iii) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \frac{x}{2}$

6. Prove that:

(i) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$

(ii) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

(iii) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

(iv) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

(v) $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\frac{1}{3}$

(vi) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(vii) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$ **[CBSE 2011, '12C]**

(viii) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\tan^{-1}\frac{4}{3}$

7. Prove that:

(i) $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$ (ii) $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}} = \frac{\pi}{2}$

(iii) $\cos^{-1}\frac{3}{5} + \sin^{-1}\frac{12}{13} = \sin^{-1}\frac{56}{65}$ (iv) $\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{27}{11}$

(v) $\tan^{-1}\frac{1}{3} + \sec^{-1}\frac{\sqrt{5}}{2} = \frac{\pi}{4}$ (vi) $\sin^{-1}\frac{1}{\sqrt{17}} + \cos^{-1}\frac{9}{\sqrt{85}} = \tan^{-1}\frac{1}{2}$

(vii) $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

8. Solve for x:

(i) $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$ **[CBSE 2008, '08C]**

(ii) $\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\frac{2}{3}$ (iii) $\cos(\sin^{-1}x)=\frac{1}{9}$

(iv) $\cos(2\sin^{-1}x)=\frac{1}{9}$ (v) $\sin^{-1}\frac{8}{x}+\sin^{-1}\frac{15}{x}=\frac{\pi}{2}$

9. Solve for x:

(i) $\cos(\sin^{-1}x)=\frac{1}{2}$ (ii) $\tan^{-1}x=\sin^{-1}\frac{1}{\sqrt{2}}$

(iii) $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6}$

ANSWERS (EXERCISE 4C)

8. (i) $x=\frac{1}{4}$ (ii) $x=3$ or $x=-3$ (iii) $x=\pm\frac{4\sqrt{5}}{9}$ (iv) $x=\pm\frac{2}{3}$ (v) $x=17$

9. (i) $x=\pm\frac{\sqrt{3}}{2}$ (ii) $x=1$ (iii) $x=\frac{\sqrt{3}}{2}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 4C)

1. (i) Put $x=\tan\theta$ (ii) LHS $=\tan^{-1}x+\frac{1}{\tan^{-1}(x+1)}$

2. Put $x=\sin 0$

3. (i) Put $x=\sin\theta$ (ii) Put $x=\cos\theta$ (iii) Put $x=\tan\theta$ (iv) Use $\tan^{-1}A+\tan^{-1}B$

4. (i) Put $x=\sin\theta$ (ii) Put $x=\cos\theta$ (iii) Put $x=\cos\theta$

(iv) Putting $x=\cot\theta$, we get:

$$\cot^{-1}(\sqrt{1+x^2}-x)=\cot^{-1}(\operatorname{cosec}\theta-\cot\theta)$$

$$=\cot^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\cot^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\cot^{-1}\left(\tan\frac{\theta}{2}\right)=\cot^{-1}\left[\cot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$$

$$=\frac{\pi}{2}-\frac{\theta}{2}=\left(\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x\right).$$

5. (i) Put $\sqrt{x}=\tan\theta$ and $\sqrt{y}=\tan\phi$ on each side.

(ii) Put $x=\tan 0$ and $\sqrt{x}=\tan\phi$ on each side.

(iii) $\text{LHS} = \tan^{-1}\left\{\dfrac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}\right\} = \tan^{-1}\left(\tan\dfrac{x}{2}\right) = \dfrac{x}{2} = \text{RHS}.$

6. (iv) $2\tan^{-1}\dfrac{1}{3} = \tan^{-1}\left\{\dfrac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right\}.$

(v) Here $2\times 1 = 2 > -1$. Now, use $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right).$

(vi) Here $2\times 3 = 6 > 1$. So, we use the formula

$$\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\left\{\frac{2+3}{1-(2\times 3)}\right\} = \pi + \tan^{-1}(-1) = \pi - \tan^{-1}1.$$

$$\therefore\quad \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi.$$

(viii) $2\left(\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9}\right) = 2\tan^{-1}\left(\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) = 2\tan^{-1}\dfrac{1}{2}$

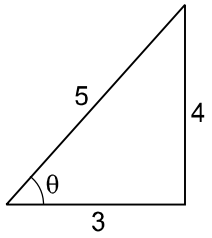
$$= \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}\right) = \tan^{-1}\frac{4}{3}.$$

7. (iii) Let $\cos^{-1}\dfrac{3}{5} = \theta.$

Then, base = 3, hypotenuse = 5.

So, perpendicular $= \sqrt{(5)^2 - 3^2} = \sqrt{16} = 4.$

$$\therefore\quad \sin\theta = \frac{4}{5} \Rightarrow \quad \theta = \sin^{-1}\frac{4}{5}.$$

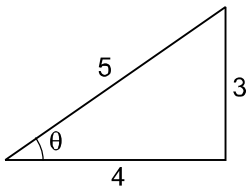


(iv) Let $\cos^{-1}\dfrac{4}{5} = \theta.$

Then, base = 4, hypotenuse = 5.

$$\therefore\quad \text{perpendicular} = \sqrt{5^2 - 4^2} = \sqrt{9} = 3.$$

$$\therefore\quad \tan\theta = \frac{4}{3} \Rightarrow\ \theta = \tan^{-1}\frac{3}{4}.$$

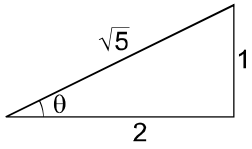


(v) Let $\sec^{-1}\dfrac{\sqrt{5}}{2} = \theta.$ Then, $\sec\theta = \dfrac{\sqrt{5}}{2}.$

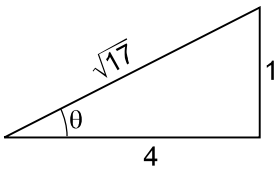
So, hypotenuse $= \sqrt{5}$, base = 2.

So, perpendicular $= \sqrt{(\sqrt{5})^2 - 2^2} = 1.$

$$\therefore\quad \tan\theta = \frac{1}{2} \Rightarrow\ \theta = \tan^{-1}\frac{1}{2}.$$



(vi) Let $\sin^{-1}\frac{1}{\sqrt{4}} = \theta$. Then, $\sin\theta = \frac{1}{\sqrt{17}}$.

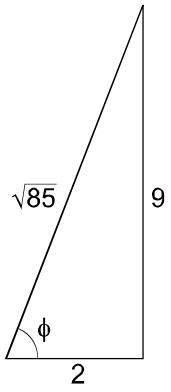


$\therefore$ perp. = 1 and hyp. = $\sqrt{17}$.

$\therefore$ base = $\sqrt{(\sqrt{17})^2 - 1^2} = \sqrt{16} = 4$.

$\therefore$ $\tan\theta = \frac{1}{4} \Rightarrow \theta = \tan^{-1}\frac{1}{4}$.

Again, let $\cos^{-1}\frac{9}{\sqrt{85}} = \phi$. Then, $\cos\phi = \frac{9}{\sqrt{85}}$.



$\therefore$ base = 9, hyp. = $\sqrt{85}$.

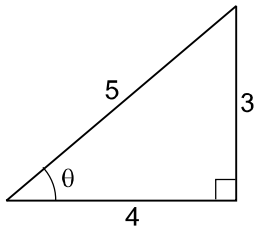
So, perp. = $\sqrt{(\sqrt{85})^2 - 9^2} = \sqrt{4} = 2$.

$\therefore$ $\tan\phi = \frac{2}{9} \Rightarrow \phi = \tan^{-1}\frac{2}{9}$.

(vii) $\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$.

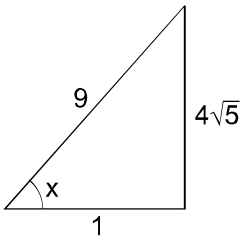
$$2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right)$$

$$= \tan^{-1}\frac{24}{7}.$$



8. (iii) $\sin^{-1}x = \cos^{-1}\frac{1}{9}$

$$= \sin^{-1}\frac{4\sqrt{5}}{9}.$$



9. (i) $\cos(\sin^{-1}x) = \frac{1}{2} = \cos\frac{\pi}{3}$ or $\cos\frac{5\pi}{3}$ $\left[\text{i.e., } \cos\left(2\pi - \frac{\pi}{3}\right)\right]$

$\Rightarrow$ $\sin^{-1}x = \frac{\pi}{3}$ or $\sin^{-1}x = \frac{5\pi}{3}$

$\Rightarrow$ $x = \sin\frac{\pi}{3}$ or $x = \sin\frac{5\pi}{3} \Rightarrow x = \frac{\sqrt{3}}{2}$ or $\frac{-\sqrt{3}}{2}$.

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

1. Graph of $\sin^{-1}x$

Let $f : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] : f(x) = \sin^{-1}x$.

Here $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal-value branch* of $\sin^{-1}x$.

The other branches of $\sin^{-1}x$ are

$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right], \ldots$ and $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-5\pi}{2}, \frac{-3\pi}{2}\right]$, etc.

Table for $\sin^{-1}x$

x	0	$\frac{1}{2}=0.5$	$\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$ $=\frac{\sqrt{2}}{2}=\frac{1.41}{2}=0.70$	$\frac{\sqrt{3}}{2}=\frac{1.73}{2}=0.86$	1
$\sin^{-1}x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Also, $\sin^{-1}(-x) = -\ \sin^{-1}x$.

$\therefore \left(x=\frac{-1}{2} \Rightarrow \sin^{-1}x=\frac{-\pi}{6}\right)$,

$\left(x=-0.7 \Rightarrow \sin^{-1}x=\frac{-\pi}{4}\right)$,

$\left(x=-0.86 \Rightarrow \sin^{-1}x=\frac{-\pi}{3}\right)$,

$\left(x=-1 \Rightarrow \sin^{-1}x=\frac{-\pi}{2}\right)$.

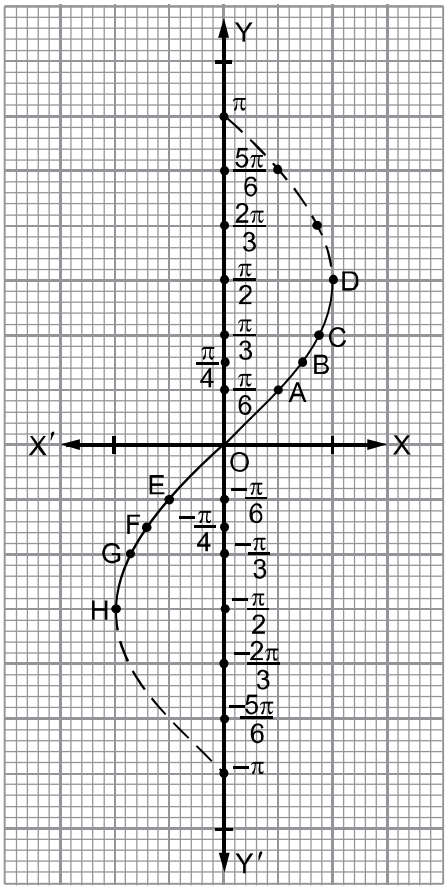
On a graph paper, we plot the points

$O(0, 0)$, $A\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $B\left(0.7, \frac{\pi}{4}\right)$,

$C\left(0.86, \frac{\pi}{3}\right)$, $D\left(1, \frac{\pi}{2}\right)$, $E\left(\frac{-1}{2}, \frac{-\pi}{6}\right)$,

$F\left(-0.7, \frac{-\pi}{4}\right)$, $G\left(-0.86, \frac{-\pi}{3}\right)$ and

$H\left(-1, \frac{-\pi}{2}\right)$.



Graph of $\sin^{-1}x$

Scale: Along the *x*-axis, 10 small divisions = 1
Along the *y*-axis, 5 small divisions = π/6

Join the points OA, AB, BC, CD, and OE, EF, FG, GH successively with a freehand to get the required graph, as shown in the given figure.

Moreover, we have

$$\sin\frac{2\pi}{3}=\sin\left(\pi-\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=0.86.$$

$$\sin\frac{5\pi}{6}=\sin\left(\pi-\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=0.5,\ \sin\pi=0.$$

$$\sin\left(\frac{-2\pi}{3}\right)=-\sin\frac{2\pi}{3}=-0.86.$$

$$\sin\left(\frac{-5\pi}{6}\right)=-\sin\frac{5\pi}{6}=-0.5.$$

Now, we may extend the graph as shown in the figure.

2. Graph of $\cos^{-1}x$

Let $f : [-1, 1] \to [0, \pi] : f(x) = \cos^{-1} x$.

Here, $[0, \pi]$ is called the principal-value branch of $\cos^{-1} x$.

The other branches of $\cos^{-1} x$ are $[\pi, 2\pi]$, $[2\pi, 3\pi]$, … , $[-\pi, 0]$, $[-2\pi, -\pi]$,etc.

Table for $\cos^{-1}x$

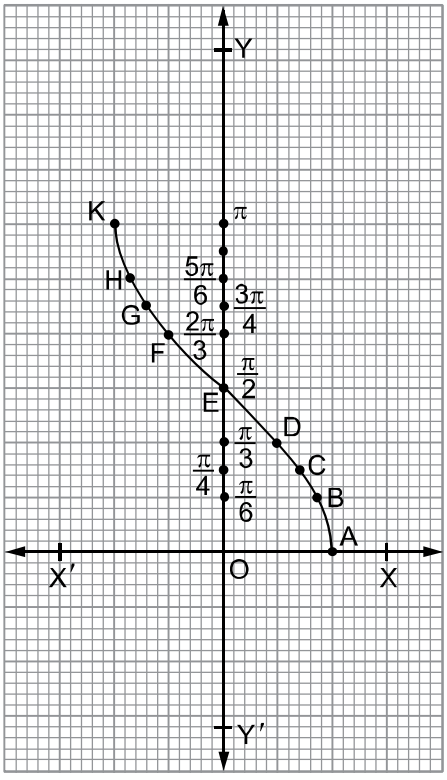
x	1	$\frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.87$	$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.41}{2} = 0.7$	$\frac{1}{2} = 0.5$	0	$\frac{-1}{2} = -0.5$	$\frac{-1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2} = \frac{-1.41}{2} = -0.70$	$\frac{-\sqrt{3}}{2} = \frac{-1.73}{2} = -0.86$	–1
$\cos^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

On a graph paper, we plot the points

$A(1, 0)$, $B\left(0.87, \frac{\pi}{6}\right)$, $C\left(0.7, \frac{\pi}{4}\right)$, $D\left(0.5, \frac{\pi}{3}\right)$, $E\left(0, \frac{\pi}{2}\right)$, $F\left(-0.5, \frac{2\pi}{3}\right)$,

$G\left(-0.7, \frac{3\pi}{4}\right)$, $H\left(-0.86, \frac{5\pi}{6}\right)$ and $K(-1, \pi)$.

Join AB, BC, CD, DE, EF, FG, GH and HK successively freehand to obtain the graph of $\cos^{-1} x$, as shown in the given figure.



Graph of $\cos^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

3. Graph of $\tan^{-1}x$

Let $f : R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) : f(x) = \tan^{-1} x.$

Here $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is the principal-value branch of $\tan^{-1} x$.

The other branches of $\tan^{-1} x$ are $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \ldots, \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, etc.

We take the positive values and then use $\tan^{-1}(-x) = -\tan^{-1} x$.

Table for $\tan^{-1} x$

x	0	$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.73}{3} = 0.58$	1	$\sqrt{3} = 1.73$
$\tan^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

Also, $\tan^{-1}(-x) = -\tan^{-1} x$.

$\therefore \quad \tan^{-1}(-0.58) = -\frac{\pi}{6}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \tan^{-1}(-1.73) = \frac{-\pi}{3}.$

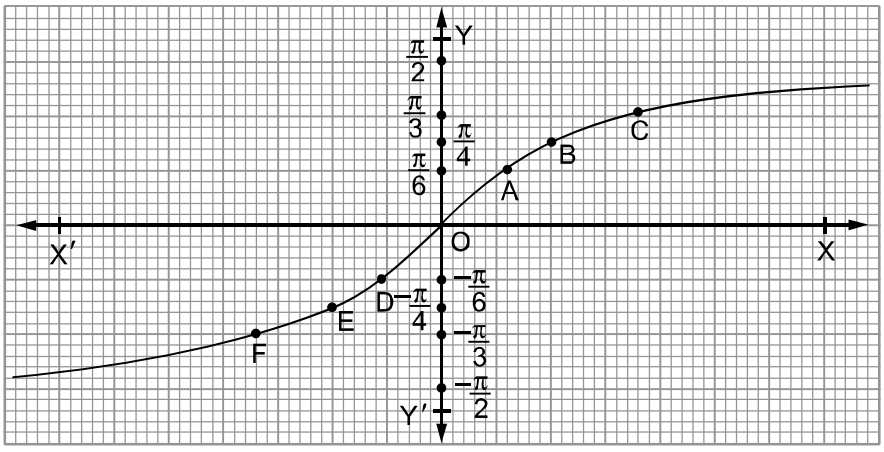
On a graph paper, we plot the points $O(0, 0), A\left(0.58, \frac{\pi}{6}\right), B\left(1, \frac{\pi}{4}\right),$

$C\left(1.73, \frac{\pi}{3}\right)$, and $D\left(-0.58, \frac{-\pi}{6}\right), E\left(-1, \frac{-\pi}{4}\right), F\left(-1.73, \frac{-\pi}{3}\right).$

Join OA, AB, BC, and OD, DE, EF successively freehand to get the graph.

Now, when $x \to \infty$, then $\tan^{-1} x \to \frac{\pi}{2}$.

Also, when $x \to -\infty$, then $\tan^{-1} x \to \frac{-\pi}{2}$.



Graph of $\tan^{-1} x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

4. Graph of $\cot^{-1}x$

Let $f : R \to (0, \pi) : f(x) = \cot^{-1} x$.

Here, $(0, \pi)$ is the principal-value branch of $\cot^{-1} x$.

The other branches of $\cot^{-1} x$ are $(\pi, 2\pi), (2\pi, 3\pi), \ldots, (-\pi, 0)$, etc.

Table for $\cot^{-1} x$

x	$\sqrt{3} = 1.73$	1	$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.732}{3} = 0.58$	0
$\cot^{-1} x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Using $\cot(\pi - x) = -\cot x$, we have

$$\cot\frac{2\pi}{3} = \cot\left(\pi - \frac{\pi}{3}\right) = -\cot\frac{\pi}{3} = -0.58, \cot\frac{3\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1,$$

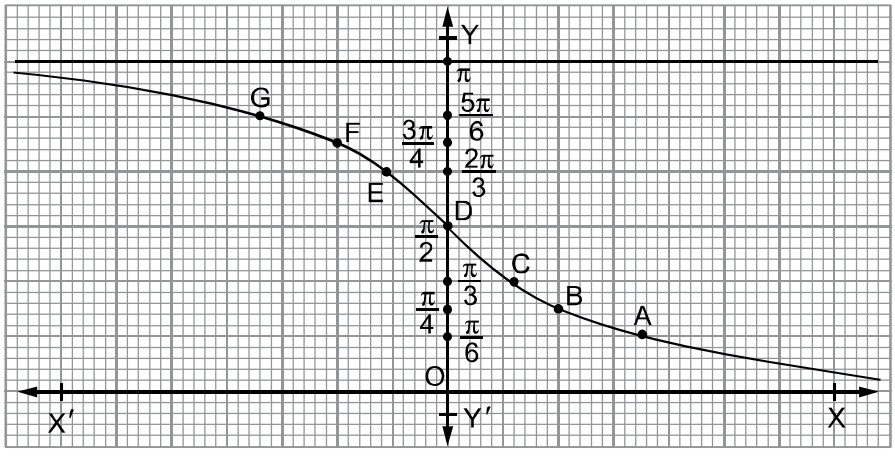
$$\cot\frac{5\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = -1.73.$$

On a graph paper, we plot the points $A\left(1.73, \frac{\pi}{6}\right)$, $B\left(1, \frac{\pi}{4}\right)$, $C\left(0.58, \frac{\pi}{3}\right)$, $D\left(0, \frac{\pi}{2}\right)$, $E\left(-0.58, \frac{2\pi}{3}\right)$, $F\left(-1, \frac{3\pi}{4}\right)$, and $G\left(-1.73, \frac{5\pi}{6}\right)$.

We join the points AB, BC, CD, DE, EF, FG successively to get the graph.

As $x \to \infty$, then $\cot^{-1} x \to 0$.

And, as $x \to -\infty$, then $\cot^{-1} x \to \pi$.



Graph of $\cot^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

5. Graph of $\sec^{-1}x$

Let $f : R - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\} : f(x) = \sec^{-1}x$.

The other branches of $\sec^{-1}x$ are $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}, \ldots, [-\pi, 0] - \left\{\frac{-\pi}{2}\right\}$, etc.

Table for $\sec^{-1}x$

x	1	$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2 \times 1.73}{3} = \frac{3.46}{3} = 1.15$	$\sqrt{2} = 1.41$	2
$\sec^{-1}x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

$\left\{\sec\frac{2\pi}{3} = -2 \Rightarrow \sec^{-1}(-2) = \frac{2\pi}{3}\right\}, \left\{\sec\frac{3\pi}{4} = -\sqrt{2} = -1.41 \Rightarrow \sec^{-1}(-1.41) = \frac{3\pi}{4}\right\},$

$\left\{\sec\frac{5\pi}{6} = \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-2\sqrt{3}}{3} = \frac{-2 \times 1.73}{3} = \frac{-3.46}{3} = -1.15 \Rightarrow \sec^{-1}(-1.15) = \frac{5\pi}{6}\right\},$

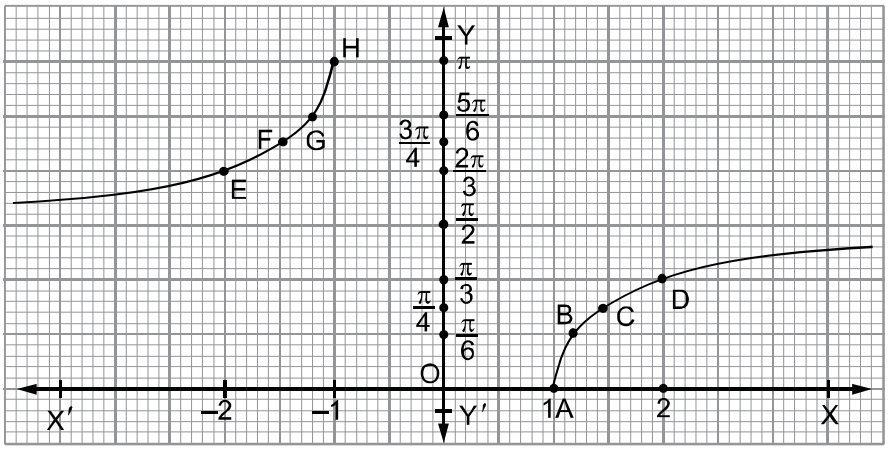
$\sec \pi = -1 \Rightarrow \sec^{-1}(-1) = \pi.$

On a graph paper, we plot the points

$A(1, 0), B\left(1.15, \frac{\pi}{6}\right), C\left(1.41, \frac{\pi}{4}\right), D\left(2, \frac{\pi}{3}\right)$ and

$E\left(-2, \frac{2\pi}{3}\right), F\left(-1.41, \frac{3\pi}{4}\right), G\left(-1.15, \frac{5\pi}{6}\right), H(-1, \pi).$

We join the points AB, BC, CD, DE, and HG, GF, FE.



Graph of $\sec^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

6. Graph of $\text{cosec}^{-1}x$

Let $f : R - (-1, 1) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\} : f(x) = \text{cosec}^{-1}x.$

The other branches of $\text{cosec}^{-1}x$ are $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$, ..., etc.

Table for $\text{cosec}^{-1}x$

x	2	$\sqrt{2} = 1.41$	$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2 \times 1.73}{3} = 1.15$	1
$\text{cosec}^{-1}x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

Since $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, we have

$$\text{cosec}^{-1}\left(\frac{-\pi}{6}\right) = -2, \text{cosec}^{-1}\left(\frac{-\pi}{4}\right) = -1.41, \text{cosec}^{-1}\left(\frac{-\pi}{3}\right) = -1.15 \text{ and}$$

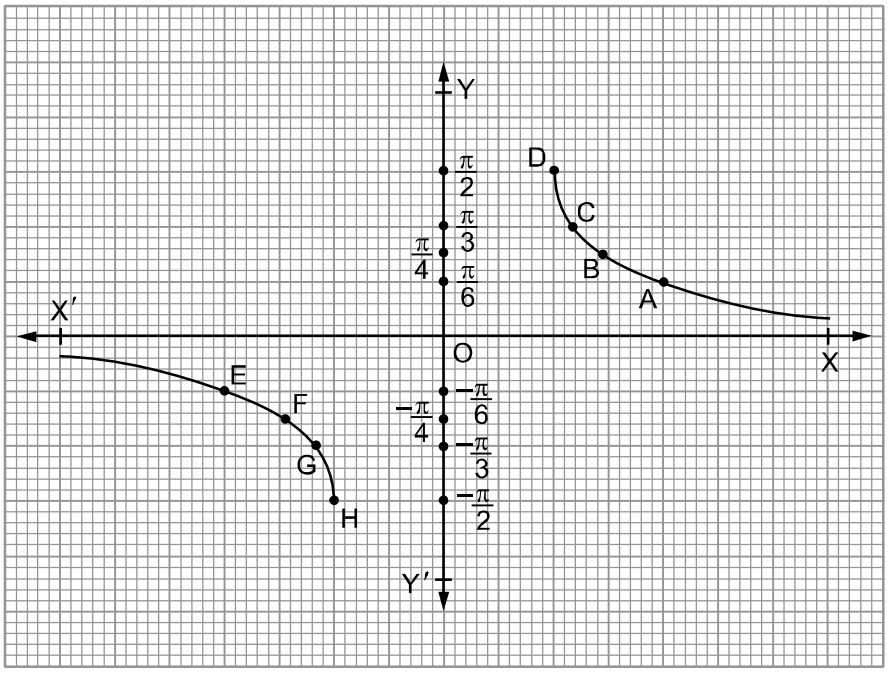
$$\text{cosec}^{-1}\left(\frac{-\pi}{2}\right) = -1.$$

On a graph paper, we plot the points $A\left(2, \frac{\pi}{6}\right)$, $B\left(1.41, \frac{\pi}{4}\right)$, $C\left(1.15, \frac{\pi}{3}\right)$, $D\left(1, \frac{\pi}{2}\right)$ and $E\left(-2, \frac{-\pi}{6}\right)$, $F\left(-1.41, \frac{-\pi}{4}\right)$, $G\left(-1.15, \frac{-\pi}{3}\right)$, $H\left(-1, \frac{-\pi}{2}\right)$.

Join the points as shown in the given figure, to get the graph.

Also as $x \to 0$ from +ve values, then $\text{cosec}^{-1}x \to \infty$.

As $x \to 0$ from –ve values, then $\text{cosec}^{-1}x \to -\infty$.



Graph of $\text{cosec}^{-1}x$

Scale: Along the x-axis, 10 small divisions = 1

Along the y-axis, 5 small divisions = $\pi/6$

EXERCISE 4D

Write down the interval for the principal-value branch of each of the following functions and draw its graph:

1. $\sin^{-1}$
2. $\cos^{-1} x$
3. $\tan^{-1} x$
4. $\cot^{-1} x$
5. $\sec^{-1} x$
6. $\operatorname{cosec}^{-1} x$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

   (a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) none of these

2. The principal value of $\operatorname{cosec}^{-1}(2)$ is

   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

3. The principal value of $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is

   (a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$

4. The principal value of $\sin^{-1}\left(\frac{-1}{2}\right)$ is

   (a) $\frac{-\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) none of these

5. The principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is

   (a) $\frac{-\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{\pi}{3}$

6. The principal value of $\tan^{-1}(-\sqrt{3})$ is

   (a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{-\pi}{3}$ (d) none of these

7. The principal value of $\cot^{-1}(-1)$ is

   (a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) $\frac{3\pi}{4}$

8. The principal value of $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ is

   (a) $\frac{\pi}{6}$ (b) $\frac{-\pi}{6}$ (c) $\frac{5\pi}{6}$ (d) $\frac{7\pi}{6}$

9. The principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is

(a) $\frac{-\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) none of these

10. The principal value of $\cot^{-1}(-\sqrt{3})$ is

(a) $\frac{-\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) $\frac{5\pi}{6}$

11. The value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{3}$ (c) $\frac{\pi}{3}$ (d) none of these

12. The value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ is

(a) $\frac{13\pi}{6}$ (b) $\frac{7\pi}{6}$ (c) $\frac{5\pi}{6}$ (d) $\frac{\pi}{6}$

13. The value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ is

(a) $\frac{7\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{6}$ (d) none of these

14. The value of $\cot^{-1}\left(\cot\frac{5\pi}{4}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{-\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) none of these

15. The value of $\sec^{-1}\left(\sec\frac{8\pi}{5}\right)$ is

(a) $\frac{2\pi}{5}$ (b) $\frac{3\pi}{5}$ (c) $\frac{8\pi}{5}$ (d) none of these

16. The value of $\text{cosec}^{-1}\left(\text{cosec}\frac{4\pi}{3}\right)$ is

(a) $\frac{\pi}{3}$ (b) $\frac{-\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) none of these

17. The value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ is

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{-\pi}{4}$ (d) none of these

18. $\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right) = ?$

(a) 0 (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

19. The value of $\sin\left\{\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right\}=?$

(a) 0 (b) 1 (c) –1 (d) none of these

20. If $x \neq 0$ then $\cos(\tan^{-1}x + \cot^{-1}x) = ?$

(a) –1 (b) 1

(c) 0 (d) none of these

21. The value of $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{-2}{5}$ (d) none of these

22. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=?$

(a) $\frac{4\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π

23. $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{-\pi}{3}$ (c) $\frac{5\pi}{3}$ (d) none of these

24. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=?$

(a) $\frac{2\pi}{3}$ (b) $\frac{3\pi}{2}$ (c) 2π (d) none of these

25. $\tan^{-1}1+\cos^{-1}\left(\frac{-1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)=?$

(a) π (b) $\frac{2\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{2}$

26. $\tan\left[2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right]=?$

(a) $\frac{7}{17}$ (b) $\frac{-7}{17}$ (c) $\frac{7}{12}$ (d) $\frac{-7}{12}$

27. $\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right)=?$

(a) $\frac{(3-\sqrt{5})}{2}$ (b) $\frac{(3+\sqrt{5})}{2}$ (c) $\frac{(5-\sqrt{3})}{2}$ (d) $\frac{(5+\sqrt{3})}{2}$

28. $\sin\left(\cos^{-1}\frac{3}{5}\right)=?$

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) none of these

29. $\cos\left(\tan^{-1}\frac{3}{4}\right) = ?$

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{4}{9}$ (d) none of these

30. $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right\} = ?$

(a) 1 (b) 0 (c) $\frac{-1}{2}$ (d) none of these

31. $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = ?$

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\frac{2}{\sqrt{10}}$

32. $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$

33. If $\cot^{-1}\left(\frac{-1}{5}\right) = x$ then $\sin x = ?$

(a) $\frac{1}{\sqrt{26}}$ (b) $\frac{5}{\sqrt{26}}$ (c) $\frac{1}{\sqrt{24}}$ (d) none of these

34. $\sin^{-1}\left(\frac{-1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = ?$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) none of these

35. $\tan^{-1}(-1) + \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = ?$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) $\frac{2\pi}{3}$

36. $\cot(\tan^{-1}x + \cot^{-1}x) = ?$

(a) 1 (b) $\frac{1}{2}$ (c) 0 (d) none of these

37. $\tan^{-1}1 + \tan^{-1}\frac{1}{3} = ?$

(a) $\tan^{-1}\frac{4}{3}$ (b) $\tan^{-1}\frac{2}{3}$ (c) $\tan^{-1}2$ (d) $\tan^{-1}3$

38. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

39. $2\tan^{-1}\frac{1}{3} = ?$

(a) $\tan^{-1}\frac{3}{2}$ (b) $\tan^{-1}\frac{3}{4}$ (c) $\tan^{-1}\frac{4}{3}$ (d) none of these

40. $\cos\left\{2\tan^{-1}\frac{1}{2}\right\} = ?$

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{7}{8}$ (d) none of these

41. $\sin\left[2\tan^{-1}\frac{5}{8}\right]$

(a) $\frac{25}{64}$ (b) $\frac{80}{89}$ (c) $\frac{75}{128}$ (d) none of these

42. $\sin\left[2\sin^{-1}\frac{4}{5}\right]$

(a) $\frac{12}{25}$ (b) $\frac{16}{25}$ (c) $\frac{24}{25}$ (d) none of these

43. If $\tan^{-1} x = \frac{\pi}{4} - \tan^{-1}\frac{1}{3}$ then $x = ?$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{6}$ (d) none of these

44. If $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \frac{\pi}{2}$ then $x = ?$

(a) 1 (b) –1 (c) 0 (d) $\frac{1}{2}$

45. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$ then $(\cos^{-1} x + \cos^{-1} y) = ?$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{2\pi}{3}$

46. $(\tan^{-1} 2 + \tan^{-1} 3) = ?$

(a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) π

47. If $\tan^{-1} x + \tan^{-1} 3 = \tan^{-1} 8$ then $x = ?$

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) 3 (d) 5

48. If $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$ then $x = ?$

(a) $\frac{1}{2}$ or –2 (b) $\frac{1}{3}$ or –3 (c) $\frac{1}{4}$ or –2 (d) $\frac{1}{6}$ or –1

49. $\tan\left\{\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right\}=?$

(a) $\frac{13}{6}$ (b) $\frac{17}{6}$ (c) $\frac{19}{6}$ (d) $\frac{23}{6}$

50. $\cot^{-1}9+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=?$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$

51. Range of $\sin^{-1}x$ is

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $[0,\pi]$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) none of these

52. Range of $\cos^{-1}x$ is

(a) $[0,\pi]$ (b) $\left[0,\frac{\pi}{2}\right]$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) none of these

53. Range of $\tan^{-1}x$ is

(a) $\left(0,\frac{\pi}{2}\right)$ (b) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (d) none of these

54. Range of $\sec^{-1}x$ is

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $[0,\pi]$ (c) $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ (d) none of these

55. Range of $\operatorname{cosec}^{-1}x$ is

(a) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (b) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (c) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ (d) none of these

56. Domain of $\cos^{-1}x$ is

(a) $[0,1]$ (b) $[-1,1]$ (c) $[-1,0]$ (d) none of these

57. Domain of $\sec^{-1}x$ is

(a) $[-1,1]$ (b) $R-\{0\}$ (c) $R-[-1,1]$ (d) $R-\{-1,1\}$

ANSWERS (OBJECTIVE QUESTIONS)

1. (a) **2.** (b) **3.** (c) **4.** (a) **5.** (b) **6.** (c) **7.** (d) **8** (c) **9.** (a) **10.** (d)
11. (c) **12.** (d) **13.** (c) **14.** (a) **15.** (a) **16.** (b) **17.** (c) **18.** (c) **19.** (b) **20.** (c)
21. (b) **22.** (d) **23.** (b) **24.** (a) **25.** (c) **26.** (b) **27.** (a) **28.** (b) **29.** (b) **30.** (a)
31. (c) **32.** (b) **33.** (b) **34.** (c) **35.** (a) **36.** (c) **37.** (c) **38.** (b) **39.** (b) **40.** (a)
41. (b) **42.** (c) **43.** (a) **44.** (c) **45.** (b) **46.** (c) **47.** (b) **48.** (d) **49.** (b) **50.** (b)
51. (c) **52.** (a) **53.** (b) **54.** (c) **55.** (c) **56.** (b) **57.** (c)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$.

2. Let $\text{cosec}^{-1}(2) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then, $\text{cosec}\, x = 2\text{cosec}\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$.

3. Let $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \frac{-1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}$.

4. Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right) \Rightarrow x = \frac{-\pi}{6}$.

5. Let $\cos^{-1}\left(\frac{-1}{2}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \Rightarrow x = \frac{2\pi}{3}$.

6. Let $\tan^{-1}(-\sqrt{3}) = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Then, $\tan x = -\sqrt{3} = -\tan\frac{\pi}{3} = \tan\left(\frac{-\pi}{3}\right) \Rightarrow x = \frac{-\pi}{3}$.

7. Let $\cot^{-1}(-1) = x$, where $x \in [0, \pi]$.

Then, $\cot x = -1 = -\cot\frac{\pi}{4} = \cot\left(\pi - \frac{\pi}{4}\right) = \cot\frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}$.

8. Let $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right) = x$, where $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Then, $\sec x = \frac{-2}{\sqrt{3}} = -\sec\frac{\pi}{6} = \sec\left(\pi - \frac{\pi}{6}\right) = \sec\frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{6}$.

9. Let $\text{cosec}^{-1}(-\sqrt{2}) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then, $\text{cosec}\, x = -\sqrt{2} = -\text{cosec}\frac{\pi}{4} = \text{cosec}\left(\frac{-\pi}{4}\right) \Rightarrow x = \frac{-\pi}{4}$.

10. Let $\cot^{-1}(-\sqrt{3}) = x$, where $x \in [0, \pi]$.

Then, $\cot x = -\sqrt{3} = -\cot\frac{\pi}{6} = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{6}$.

11. Let $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Then, $\sin x = \sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} \Rightarrow x = \frac{\pi}{3}$.

12. Let $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = x$, where $x \in [0, \pi]$.

Then, $\cos x = \cos\frac{13\pi}{6} = \cos\left(2\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$.

13. Let $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Then, $\tan x = \tan\frac{7\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right) = \tan\frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$.

14. Let $\cot^{-1}\left(\cot\frac{5\pi}{4}\right) = x$, where $x \in [0, \pi]$.

Then, $\cot x = \cot\frac{5\pi}{4} = \cot\left(\pi + \frac{\pi}{4}\right) = \cot\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$.

15. Let $\sec^{-1}\left(\sec\frac{8\pi}{5}\right) = x$, where $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Then, $\sec x = \sec\frac{8\pi}{5} = \sec\left(2\pi - \frac{2\pi}{5}\right) = \sec\frac{2\pi}{5} \Rightarrow x = \frac{2\pi}{5}$.

16. Let $\text{cosec}^{-1}\left(\text{cosec}\frac{4\pi}{3}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Then, $\text{cosec}\, x = \text{cosec}\frac{4\pi}{3} = \text{cosec}\left(\pi + \frac{\pi}{3}\right) = -\,\text{cosec}\frac{\pi}{3} = \text{cosec}\left(\frac{-\pi}{3}\right)$.

$\therefore \quad x = \frac{-\pi}{3}$.

17. Let $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

Then, $\tan x = \tan\frac{3\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\left(\frac{-\pi}{4}\right) \Rightarrow x = \frac{-\pi}{4}$.

18. Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Then, $\sin x = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right) \Rightarrow x = \frac{-\pi}{6}$.

$\therefore \quad \text{given exp.} = \frac{\pi}{3} - \left(\frac{-\pi}{6}\right) = \left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \frac{3\pi}{6} = \frac{\pi}{2}$.

19. $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2} = \frac{\pi}{2} \Rightarrow \sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right) = \sin\frac{\pi}{2} = 1$.

20. $\cos(\tan^{-1} x + \cot^{-1} x) = \cos\frac{\pi}{2} = 0$.

21. $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} \Rightarrow \cos^{-1}\frac{3}{5} = \sin^{-1}\sqrt{1-\frac{9}{25}} = \sin^{-1}\frac{4}{5}$.

$\therefore \quad \sin\left(\cos^{-1}\frac{3}{5}\right) = \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}$.

22. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$= \cos^{-1}\left\{\cos\left(\pi - \frac{\pi}{3}\right)\right\} + \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$

$= \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + \sin^{-1}\left\{\sin\frac{\pi}{3}\right\} = \cos^{-1}\left(\frac{-1}{2}\right) + \frac{\pi}{3}$

$= \left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = \pi.$

23. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \tan^{-1}\sqrt{3} - (\pi - \sec^{-1}2)$

$= \frac{\pi}{3} - \pi + \frac{\pi}{3} = \frac{-\pi}{3}$.

24. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = \frac{\pi}{3} + \left(2 \times \frac{\pi}{6}\right) = \left(\frac{\pi}{3} + \frac{\pi}{3}\right) = \frac{2\pi}{3}$.

25. $\tan^{-1} 1 + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{4} + \left(\pi - \cos^{-1}\frac{1}{2}\right) - \sin^{-1}\frac{1}{2}$

$= \left(\frac{\pi}{4} + \pi - \frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{3\pi}{4}$.

26. Let $2\tan^{-1}\frac{1}{5} = \theta$. Then, $\tan^{-1}\frac{1}{5} = \frac{1}{2}\theta \Rightarrow \tan\frac{1}{2}\theta = \frac{1}{5}$.

$\therefore \quad \tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta} = \frac{\left(2 \times \frac{1}{5}\right)}{\left(1 - \frac{1}{25}\right)} = \left(\frac{2}{5} \times \frac{25}{24}\right) = \frac{5}{12}$.

$\therefore \quad \tan\left[2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right] = \tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\theta \cdot \tan\frac{\pi}{4}}$

$= \frac{\left(\frac{5}{12} - 1\right)}{\left(1 + \frac{5}{12} \times 1\right)} = \frac{\left(\frac{-7}{12}\right)}{\left(\frac{17}{12}\right)} = \frac{-7}{17}$.

27. Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$.

$\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right) = \tan\frac{1}{2}\theta = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \left\{\frac{\left(1-\frac{\sqrt{5}}{3}\right)}{\left(1+\frac{\sqrt{5}}{3}\right)}\right\}^{1/2}$$

$$= \left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)^{1/2} = \left\{\frac{3-\sqrt{5}}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}}\right\}^{1/2} = \frac{(3-\sqrt{5})}{2}.$$

28. Let $\cos^{-1}\frac{3}{5} = x$, where $x \in [0, \pi]$. Then, $\cos x = \frac{3}{5}$.

$\therefore$ since $x \in [0, \pi]$, $\sin x > 0$.

$\therefore$ $\sin x = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \Rightarrow \sin\left(\cos^{-1}\frac{3}{5}\right) = \frac{4}{5}$.

29. Let $\tan^{-1}\frac{3}{4} = x$, where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore$ $\tan x = \frac{3}{4}$ and since $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, we have $\cos x > 0$.

$\therefore$ $\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{4}{5}$.

$\therefore$ $\cos\left(\tan^{-1}\frac{3}{4}\right) = \cos x = \frac{4}{5}$.

30. $\sin\left\{\frac{\pi}{3} - \sin^{-3}\left(\frac{-1}{2}\right)\right\} = \sin\left\{\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right\} = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1.$

31. Let $\cos^{-1}\frac{4}{5} = x$, where $x \in [0, \pi]$. Then, $\cos x = \frac{4}{5}$.

Since $x \in [0, \pi] \rightarrow \frac{1}{2}x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin\frac{1}{2}x > 0.$

$$\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{\left(1-\frac{4}{5}\right)}{2}} = \frac{1}{\sqrt{10}}.$$

32. $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = \tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\}$

$$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\} = \tan^{-1}\left\{2\times\frac{1}{2}\right\} = \tan^{-1}1 = \frac{\pi}{4}.$$

33. $\cot^{-1}\left(\frac{-1}{5}\right) = x \Rightarrow \cot x = \frac{-1}{5}$, where $x \in (0, \pi)$.

$\sin x > 0$ in $(0, \pi)$.

$$\sin x = \frac{1}{\operatorname{cosec} x} = \frac{1}{\sqrt{1+\cot^2 x}} = \frac{1}{\sqrt{1+\frac{1}{25}}} = \frac{5}{\sqrt{26}}.$$

34. Range of $\cos^{-1}$ is $[0, \pi]$.

$$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = x \Rightarrow \cos x = \frac{-\sqrt{3}}{2} = -\cos\frac{\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = \cos\frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{6}.$$

$$\sin^{-1}\left(\frac{-1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\sin^{-1}\frac{1}{2} + 2 \times \frac{5\pi}{6} = -\frac{\pi}{6} + \frac{5\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}.$$

35. Range of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\tan^{-1}(-1) = x \Rightarrow \tan x = -1 = -\tan\frac{\pi}{4} = \tan\left(\frac{-\pi}{4}\right) \Rightarrow x = \frac{-\pi}{4}$$

Range of $\cos^{-1}$ is $[0, \pi]$.

$$\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y \Rightarrow \cos y = \frac{-1}{\sqrt{2}} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4} \Rightarrow y = \frac{3\pi}{4}.$$

$$\therefore \quad \text{given exp.} = -\frac{\pi}{4} + \frac{3\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

36. Given exp. $= \cot\frac{\pi}{2} = 0.$

37. $$\tan^{-1} 1 + \tan^{-1}\frac{1}{3} = \tan^{-1}\left\{\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right\} = \tan^{-1} 2.$$

38. $$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\left\{\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}}\right\} = \tan^{-1} 1 = \frac{\pi}{4}.$$

39. Use $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$.

40. Use $2\tan^{-1} x = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$.

41. Use $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$.

42. Use $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1 - x^2}]$.

43. $$\frac{\pi}{4} - \tan^{-1}\frac{1}{3} = \tan^{-1} 1 - \tan^{1}\frac{1}{3} = \tan^{-1}\left\{\frac{\left(1 - \frac{1}{3}\right)}{\left(1 + \frac{1}{3}\right)}\right\} = \tan^{-1}\frac{1}{2} \Rightarrow x = \frac{1}{2}.$$

44. We know that $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \Rightarrow \tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}.$

$\therefore \quad \tan^{-1}(1+x)+\tan^{-1}(1-x)=\dfrac{\pi}{2} \Rightarrow (1-x)=\dfrac{1}{(1+x)}$

$\Rightarrow (1-x^2)=1 \Rightarrow x=0.$

45. $\sin^{-1}x+\sin^{-1}y=\dfrac{2\pi}{3} \Rightarrow \left(\dfrac{\pi}{2}-\cos^{-1}x\right)+\left(\dfrac{\pi}{2}-\cos^{-1}y\right)=\dfrac{2\pi}{3}.$

$\therefore \quad \cos^{-1}x+\cos^{-1}y=\left(\pi-\dfrac{2\pi}{3}\right)=\dfrac{\pi}{3}.$

46. $(x=2, y=3) \Rightarrow xy>1.$

$\therefore \quad \tan^{-1}2+\tan^{-1}3=\pi+\tan^{-1}\left(\dfrac{2+3}{1-2\times 3}\right)=\pi+\tan^{-1}(-1)$

$=\pi-\tan(1)=\left(\pi-\dfrac{\pi}{4}\right)=\dfrac{3\pi}{4}.$

47. $\tan^{-1}x+\tan^{-1}3=\tan^{-1}8 \Rightarrow \tan^{-1}\left(\dfrac{3+x}{1-3x}\right)=\tan^{-1}8.$

$\therefore \quad \dfrac{3+x}{1-3x}=8 \Rightarrow 3+x=8-24x \Rightarrow x=\dfrac{1}{5}.$

48. $\tan^{-1}3x+\tan^{-1}2x=\dfrac{\pi}{4} \Rightarrow \tan^{-1}\left(\dfrac{3x+2x}{1-6x^2}\right)=\dfrac{\pi}{4}$

$\therefore \quad \dfrac{5x}{1-6x^2}=\tan\dfrac{\pi}{4}=1 \Rightarrow 6x^2+5x-1=0$

$\Rightarrow (x+1)(6x-1)=0$

$\Rightarrow x=-1 \text{ or } x=\dfrac{1}{6}.$

49. $\cos^{-1}x=\tan^{-1}\dfrac{\sqrt{1-x^2}}{x} \Rightarrow \cos^{-1}\dfrac{4}{5}=\tan^{-1}\dfrac{\sqrt{1-\frac{16}{25}}}{\left(\frac{4}{5}\right)}=\tan^{-1}\dfrac{3}{4}$

$\therefore \quad \cos^{-1}\dfrac{4}{5}+\tan^{-1}\dfrac{2}{3}=\tan^{-1}\dfrac{3}{4}+\tan^{-1}\dfrac{2}{3}=\tan^{-1}\dfrac{\left(\frac{3}{4}+\frac{2}{3}\right)}{\left(1-\frac{3}{4}\times\frac{2}{3}\right)}=\tan^{-1}\dfrac{17}{6}$

$\therefore \quad \text{given exp.}=\tan\left\{\tan^{-1}\dfrac{17}{6}\right\}=\dfrac{17}{6}.$

50. $\operatorname{cosec}^{-1}x=\cot^{-1}\sqrt{x^2-1} \Rightarrow \operatorname{cosec}^{-1}\dfrac{\sqrt{41}}{4}=\cot^{-1}\sqrt{\dfrac{41}{16}-1}=\cot^{-1}\dfrac{5}{4}$

$\therefore \quad \cot^{-1}9+\cot^{-1}\dfrac{5}{4}=\tan^{-1}\dfrac{1}{9}+\tan^{-1}\dfrac{4}{5}=\tan^{-1}\left(\dfrac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\times\frac{4}{5}}\right)=\tan^{-1}1=\dfrac{\pi}{4}.$